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TITLE: RADIATION HYDRODYNAMICS

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PREFACE

On March 22 through March 26, 1982 I gave a short course at Los Alamos on radiation hydrodynamics. The notes which constitute this report were prepared for use in that course. The course consisted of 14 unclassified lecture hours, and two separate classified discussion sessions. The unclassified lectures were videotaped and these tapes are available for viewing through the Los Alamos Traning Office.

Much of the material in these notes was taken from my book: The Equations of Radiation Hydrodynamics, Pergamon Press, Oxford, 1973. This book was distributed to the persons attending the course. References for this material as well as general reading references can be found in the book. Some new material not found in the book was included in the class and in these notes. References for this material are given at the end of the notes.

It is a pleasure to acknowledge the hospitality of the Laboratory, and the help of Gloria Cordova of the training office in arranging this course. Keith Taggart (X-7) very kindly offered to arrange for the typing of these notes, and Bob Weaver (X-7) undertook the task of arranging the classified discussion sessions. A very special thanks is due Tessa Lippiatt. She delivered on Keith's offer and prepared this typed version of the notes. As is evident, she did a beautiful job. I offered her a job at UCLA, but she prefers the clean air of Los Alamos.

G. C. Pomraning
September, 1982

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ABSTRACT OF COURSE

This course was intended to provid the participant with an introduction to the theory of radiative transfer, and an understanding of the coupling of radiative processes to the equations describing compressible flow. At moderate temperatures (thousands of degrees), the role of the radiation is primarily one of transporting energy by radiative processes. At higher temperatures (millions of degrees), the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities. In this case, the radiation field significantly affects the dynamics of the fluid, and it is the description of this regime which is generally the charter of "radiation hydrodynamics". The course provided a discussion of the relevant physics and a derivation of the corresponding equations, as well as an examination of several simplified models.

RADIATION - HYDRODYNAMICS

Los Alamos National Laboratory Short Course

March 22 - March 26, 1982

by

G. C. Pomraning

I. INTRODUCTION

We will be concerned with the propagation of thermal radiation through a fluid, and the effect of this radiation on the hydrodynamics describing the fluid motion. The term "thermal radiation" means electromagnetic radiation of atomic, as opposed to nuclear origin. Such radiation is generally emitted by matter in a state of thermal excitation, thus accounting for the designation of the radiation as thermal. The energy density of this type of radiation in an enclosure whose walls are maintained at a constant and uniform temperature is given by the well-known Planck formula. More generally, however, the energy distribution of the radiation field is not described by the Planck function. Under certain rather unrestrictive conditions, the state of the radiation can be described by a kinetic (transport) equation; referred to historically as the equation of radiative transfer. This introduction to radiation-hydrodynamics will, in large part, concentrate on various formulations of the equation of transfer describing the propagation of thermal radiation.

The importance of thermal radiation in physical problems increases as the temperature is raised, primarily because the radiation energy density associated with a Planck distribution varies as the fourth power of the temperature. At low temperatures (say, room temperature) radiation can generally be neglected entirely in most problems. At moderate temperatures (say thousands of degrees) the role of radiation is primarily one of transporting energy by radiative processes. At higher temperatures (say, millions of degrees) the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities. In this case, the radiation field significantly affects the dynamics of the fluid. Hydrodynamics with explicit account of the radiation energy and momentum contributions constitutes the charter of radiation-hydrodynamics. Such considerations find their practical application in the understanding of certain astrophysical and nuclear weapons effects phenomena.

These notes are roughly divided into four major topics:

1. Introductory Material, which includes a summary of the basic fluid dynamics equations without radiative contributions, an introduction to the radiation field and its interaction with matter, and the fluid equations in the presence of radiation.

2. The Equation of Radiation Transfer, which includes both an Eulerian and Lagrangian derivation, boundary and initial conditions, specific geometry representations, an integral equation formulation, Peierls' equation, induced processes, the concept of local thermodynamic equilibrium, Kirchoff's law, transport in a vacuum, and relativistic corrections.

3. Approximate Models of Radiative Transfer, such as the Eddington (diffusion) approximation, asymptotic diffusion theory, variable Eddington factors and flux limiters, equilibrium diffusion theory, Marshak waves, the spherical harmonic (P-N) method, the discrete ordinate (S-N) method, the Monte Carlo method, the formal solution (methods of characteristics), and the multi-group method, introducing the Planck and Rosseland means.

4. The Interaction of Radiation With Matter, including a discussion of the absorption coefficient, Compton and inverse Compton scattering, and the Fokker-Planck treatment of scattering.

A. The Fluid Equations Without Radiation

The non-relativistic, ideal fluid equations are, in the Eulerian conservative form:

Conservation of mass (continuity)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \quad , \tag{1}$$

Conservation of momentum (force balance)

$$\frac{\partial (\rho \vec{u})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}\vec{u}) + \vec{\nabla} P_m = 0 \quad , \tag{2}$$

Conservation of energy

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 + E_m\right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2} \rho u^2 + E_m + P_m\right)\vec{u}\right] = 0 \quad , \tag{3}$$

where

ρ = fluid density
$\vec{u}$ = fluid velocity
P_m = fluid (material) pressure
E_m = fluid (material) internal energy density.

These three equations are supplemented by an equation of state such as

$$P_m = P_m(\rho, T) \quad , \tag{4}$$

and a thermodynamic expression for the internal energy such as

$$E_m = E_m(\rho, T) \quad , \tag{5}$$

where

$$T = \text{fluid temperature} \quad . \tag{6}$$

Equations (1) through (5) represent seven equations for the seven unknowns:

$$\rho, \ \vec{u} \ \text{(three components)}, \ P_m, \ E_m, \ T \quad .$$

Note that the right hand sides of Eqs. (1) through (3) will not be zero if external sources of mass, momentum, and energy are present.

B. The Fluid Equations With Radiation

If radiation is important (i.e., if the temperature is high enough), these balance equations need be modified to include the radiation contributions. We define

E = radiation energy density
$\vec{F}$ = radiative flux of energy
$\vec{M}_d$ = radiation momentum density
$\overleftrightarrow{M}_f$ = radiative flux of momentum

The balance equations for momentum and energy, Eqs. (2) and (3), then become

$$\frac{\partial}{\partial t} (\rho\vec{u} + \vec{M}_d) + \vec{\nabla}\cdot(\rho\vec{u}\vec{u} + \overleftrightarrow{M}_f) + \vec{\nabla}P_m = 0 \quad , \tag{7}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + E_m + E\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + E_m + P_m\right)\vec{u} + \vec{F}\right] = 0 \quad . \quad (8)$$

We have now introduced several new dependent variables, namely E, $\vec{F}$, $\vec{M}_d$, and $\vec{\vec{M}}_f$. We obviously need more equations. We now show that these new variables are simply angular moments of a distribution function of the radiation field.

C. The Radiation Field

We consider the radiation field to consist of point, massless particles called photons. (We discuss the validity of this description later). With each photon we associate a frequency ν such that the energy of a photon is $h\nu$, where h is Planck's constant. It is known that such a massless particle carries momentum of magnitude $h\nu/c$, where c is the vacuum speed of light.

At any time t, six variables are required to specify the position of the photon in phase space, namely three position variables and three momentum variables. We denote the three position variables by the vector $\vec{r}$. In radiative transfer work it is conventional to use, rather than the three momentum variables, three equivalent variables. These are the frequency ν and the direction of travel of the photon $\vec{\Omega}$. We then define the distribution function f

$$f = f(\vec{r},\nu,\vec{\Omega},t) \quad , \quad (9)$$

such that

$$dn = f d\vec{r} d\nu d\vec{\Omega} \quad , \quad (10)$$

where dn is the number of photons (at time t) at $\vec{r},\nu,\vec{\Omega}$, in the six-dimensional differential volume $d\vec{r}d\nu d\vec{\Omega}$. In radiative transfer, it is conventional to introduce the specific intensity of radiation, defined as

$$I(\vec{r},\nu,\vec{\Omega},t) \equiv ch\nu f(\vec{r},\nu,\vec{\Omega},t) \quad . \tag{11}$$

In terms of f (or I), we can compute the radiative terms in the fluid equations. We have

$$E = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}(h\nu)f = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega} I \quad , \tag{12}$$

$$\vec{F} = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}(c\vec{\Omega})(h\nu)f = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega} I \quad , \tag{13}$$

$$\vec{M}_d = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\left(\frac{h\nu\vec{\Omega}}{c}\right)f = \frac{1}{c^2}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega} I = \frac{1}{c^2}\vec{F} \quad , \tag{14}$$

$$\vec{\vec{M}}_f = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}(c\vec{\Omega})\left(\frac{h\nu\vec{\Omega}}{c}\right)f = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega}\vec{\Omega} I = \vec{\vec{P}} \quad . \tag{15}$$

where the <u>radiation pressure</u>, $\vec{\vec{P}}$, is defined by the last equality in Eq. (15). Hence the nonrelativistic ideal radiation-hydrodynamic equations are:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \quad , \tag{16}$$

$$\frac{\partial}{\partial t}\left(\rho\vec{u} + \frac{1}{c^2}\vec{F}\right) + \vec{\nabla}P_m + \vec{\nabla}\cdot(\rho\vec{u}\vec{u} + \vec{\vec{P}}) = 0 \quad , \tag{17}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + E_m + E\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + E_m + P_m\right)\vec{u} + \vec{F}\right] = 0 \quad , \tag{18}$$

which are supplemented by the thermodynamic relationships

$$P_m = P_m\ (\rho,T) \quad , \tag{19}$$

$$E_m = E_m\ (\rho,T) \quad , \tag{20}$$

and the definitions of E, $\vec{F}$, and $\vec{\vec{P}}$ as

$$E = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega} I \quad , \tag{21}$$

$$\vec{F} = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega} I \quad , \tag{22}$$

$$\vec{\vec{P}} = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\vec{\Omega}\vec{\Omega} I \quad . \tag{23}$$

Thus, the inclusion of radiation in the fluid equations has introduced one new dependent variable, namely $I(\vec{r},\nu,\vec{\Omega},t)$. We need derive an equation (a conservation equation for photons) which yields I. We do this shortly.

If I is independent of $\vec{\Omega}$, it is said to be <u>isotropic</u>. If it is independent of both $\vec{r}$ and $\vec{\Omega}$, it is said to be <u>homogeneous</u> and <u>isotropic</u>. The most important example of a homogeneous and isotropic radiation field is that which coexists with matter in complete thermodynamic equilibrium at temperature T. In this case, I is the <u>Planck function</u> $B(\nu,T)$ given by

$$I = B = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1} \quad , \tag{24}$$

where k is the Boltzmann constant. The corresponding energy density is

$$E = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\left(\frac{2h\nu^3}{c^2}\right)\left(e^{h\nu/kT} - 1\right)^{-1}$$

$$= \frac{8\pi k^4 T^4}{h^3 c^3}\int_0^\infty dx x^3 (e^x - 1)^{-1}$$

$$= \frac{8\pi k^4 T^4}{h^3 c^3}\int_0^\infty dx x^3 \sum_{n=1}^{\infty} e^{-nx} = \left(\frac{48\pi k^4}{h^3 c^3} T^4\right) \sum_{n=1}^{\infty} n^{-4} \quad ,$$

or

$$E = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4 \quad . \tag{25}$$

This is often written

$$E = aT^4 \quad , \tag{26}$$

where a, the radiation constant, is given by

$$a = \frac{8\pi^5 k^4}{15 h^3 c^3} \quad , \tag{27}$$

or

$$E = \frac{4\sigma}{c} T^4 \quad , \tag{28}$$

where $\sigma = ac/4$ is called the Stefan-Boltzmann constant. The radiative flux and pressure tensor corresponding to a Planck function are

$$\vec{F} = 0 \; ; \quad \vec{\vec{P}} = \frac{aT^4}{3} \vec{\vec{I}} \quad , \tag{29}$$

where $\overleftrightarrow{I}$ is the unit (diagonal) tensor. We note the general relationship, for any I,

$$\mathrm{Tr}(\overleftrightarrow{P}) = E \quad . \tag{30}$$

We also note that for streaming radiation (i.e., all photons going in one direction, say the z direction),

$$P_{zz} = E \quad , \tag{31}$$

and all other eight components of $\overleftrightarrow{P}$ are zero.

D. Interaction Of The Radiation Field With Matter

We consider three interactions of photons with matter: (1) absorption; (2) scattering; (3) birth in the matter.

1. Absorption. We define the macroscopic absorption coefficient, or cross section,

$$\sigma_a = \sigma_a(\vec{r},\nu,t) \quad , \tag{32}$$

such that the probability of a photon being absorbed in a distance ds is

$$\text{prob. of abs.} = \sigma_a ds \quad . \tag{33}$$

In writing Eq. (32) we have assumed no angular dependence for σ_a (i.e., the matter is isotropic - has no preferred direction). This is always true except in crystals (of no importance in radiation-hydrodynamics), or if one considers relativistic effects, which can be important. We consider relativistic effects later on. The function σ_a is often decomposed as

$$\sigma_a(\vec{r},\nu,t) = \rho(\vec{r},t)\ \kappa(\vec{r},\nu,t) \quad , \tag{34}$$

where κ is the mass absorption coefficient, or opacity. Another decomposition is

$$\sigma_a(\vec{r},\nu,t) = N(\vec{r},t)\ \mu(\vec{r},\nu,t) \quad , \tag{35}$$

where N is the atomic density and μ_a is the microscopic absorption coefficient.

2. Scattering. Similarly, we define the scattering coefficient or cross section

$$\sigma_s = \sigma_s(\vec{r},\nu,t) \quad , \tag{36}$$

such that, in a distance of travel ds,

$$\text{prob. of scatt.} = \sigma_s ds \quad . \tag{37}$$

As with absorption, σ_s is independent of $\vec{\Omega}$ (except for relativistic effects). In a scattering event a photon does not disappear as in absorption, but continues to exist with another direction of travel and frequency, in general. That is, scattering changes a photon's characteristics from ν' and $\vec{\Omega}'$ to ν and $\vec{\Omega}$. We describe this by introducing the differential scattering coefficient or the differential scattering cross $\sigma_s(\nu'\rightarrow\nu,\vec{\Omega}\cdot\vec{\Omega}')$ such that the probabilit of a photon being scattered from ν' to ν contained in $d\nu$, and f: $\vec{\Omega}'$ to $\vec{\Omega}$ contained in $d\vec{\Omega}$, in traveling a distance ds is given

$$\text{prob} = \sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega}) d\nu d\vec{\Omega} ds \quad . \tag{38}$$

Note the argument $\vec{\Omega}' \cdot \vec{\Omega}$ rather than $\vec{\Omega}'$ and $\vec{\Omega}$ separately. That is, the scattering depends upon the scattering angle only, not $\vec{\Omega}'$ and $\vec{\Omega}$ separately. This is a consequence of the assumption of isotropic matter. We also note the identity between the scattering cross section $\sigma_s(\nu')$ and the differential scattering cross section $\sigma_s(\nu' \to \nu, \vec{\Omega} \cdot \vec{\Omega}')$

$$\sigma_s(\nu') = \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega} \sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega}) \quad , \tag{39}$$

or

$$\sigma_s(\nu') = 2\pi \int_0^\infty d\nu \int_{-1}^{1} d\mu \ \sigma_s(\nu' \to \nu, \mu) \quad . \tag{40}$$

Often one decomposes $\sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega})$ as

$$\sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega}) = \sigma_s(\nu') \ K(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega}) \quad , \tag{41}$$

where K is the normalized scattering distribution, i.e.,

$$\int_0^\infty d\nu \int_{4\pi} d\vec{\Omega} \ K(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega}) = 1 \quad . \tag{42}$$

If

$$K(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega}) = K(\vec{\Omega}' \cdot \vec{\Omega}) \ \delta(\nu' - \nu) \quad , \tag{43}$$

where

$$\int_{4\pi} d\vec{\Omega}\, K(\vec{\Omega}'\cdot\vec{\Omega}) = 1 \quad , \tag{44}$$

the scattering is called <u>coherent</u> or <u>conservative</u>. If

$$K(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega}) = \frac{1}{4\pi}\, K(\nu'\to\nu) \quad , \tag{45}$$

where

$$\int_0^\infty d\nu\, K(\nu'\to\nu) = 1 \quad , \tag{46}$$

the scattering is called <u>isotropic</u>. The simplest scattering distribution is both coherent and isotropic, i.e.,

$$K(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega}) = \frac{1}{4\pi}\, \delta(\nu'-\nu) \tag{47}$$

and is widely used in radiation-hydrodynamic calculations.

<u>Total Cross Section - Mean Free Path</u>

We define the <u>total interaction coefficient or total cross section</u> σ as

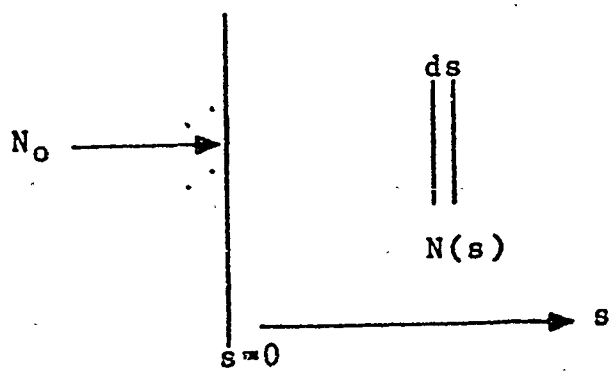
$$\sigma = \sigma_a + \sigma_s \quad . \tag{48}$$

The symbol $\tilde{\omega}$ generally denotes the probability of scattering given that a collision has occurred, i.e.,

$$\tilde{\omega} = \frac{\sigma_s}{\sigma_a + \sigma_s} = \frac{\sigma_s}{\sigma} \quad . \tag{49}$$

$\tilde{\omega} = 0$ is a purely absorbing problem, and $\tilde{\omega} = 1$ is a purely scattering one.

We ask the question: If a photon of frequency ν is traveling in a homogeneous medium (no $\vec{r}$ and t dependences) of total cross section $\sigma(\nu)$, how far, on the average, will this photon stream before suffering a collision? This distance is called the <u>mean free path</u> and denoted by $\lambda = \lambda(\nu)$.



We have

$$dN(\text{collided}) = -N\sigma ds \quad , \tag{50}$$

from the definition of the total cross section σ. Hence

$$\frac{dN}{ds} = -N\sigma \quad , \tag{51}$$

or

$$N = N_o e^{-\sigma s} \quad . \tag{52}$$

That is, N_o photons initially in a beam will be exponentially reduced to N photons in the beam after traveling a distance s. The number of photons that collide in a path length element ds at s is then given by

$$|dN| = N_o e^{-\sigma s}\sigma ds \quad . \tag{53}$$

This number of photons have traveled a distance s before suffering a collision and hence the average distance $\bar{s}$, or λ, to a collision is just s averaged over $|dN|$, i.e.,

$$\bar{s} = \lambda = \frac{\int_0^\infty s(N_o e^{-\sigma s} \sigma ds)}{\int_0^\infty (N_o e^{-\sigma s} \sigma ds)} = \frac{1}{\sigma} \quad , \tag{54}$$

or, displaying, the frequency argument,

$$\lambda(\nu) = \frac{1}{\sigma(\nu)} \quad . \tag{55}$$

That is, the average distance a photon travels between collisions is just the inverse of the total cross section (in a homogeneous medium).

3. Birth. Consider the emission of photons. Since neither absorption nor scattering creates photons, how are photons introduced into a system? One way this can be accomplished is by shining light into the matter through its bounding surface. We discuss this shortly.

The other possibility is that photons are born in the matter, through the process of spontaneous emission. That is, all materials spontaneously emit photons characteristic of the state of the matter. We quantify this source by introducing

$$q = q(\vec{r}, \nu, t) \quad , \tag{56}$$

such that the number of photons emitted per unit time and volume at frequency ν in $d\nu$ and direction $\vec{\Omega}$ in $d\vec{\Omega}$ is given by

$$\text{photons emitted} = q(\vec{r}, \nu, t) d\nu d\vec{\Omega} \quad . \tag{57}$$

This source of photons is taken to be independent of $\vec{\Omega}$, which again follows from the assumption of isotropic matter.

II. THE EQUATION OF TRANSFER

We derive an equation satisfied by $I(\vec{r},\nu,\vec{\Omega},t)$, the equation of radiative transfer. This is just a conservation equation for photons.

A. Assumptions and Limitations

In order to obtain a relatively simple kinetic (transport) equation we need to approximate the underlying physics of radiation processes. These approximations fall into two classes - those that are inherent in any kinetic equation description of radiation energy transport, and those that can be incorporated into such an equation at the expenses of simplicity. It should be emphasized that the question of the inherent validity of a kinetic equation for photons is by no means settled, but is still being actively researched.

Inherent Limitations

1. Photon density is large, so that fluctuations caused by individual photon dynamics can be ignored - it suffices to deal with averages, as is inherent in characterizing the photon distribution in a statistical way with a single particle distribution function.

2. The wave packet we call a photon is small, in both physical and momentum space. That is, the spreads in these variables must be small compared to the resolution of interest in

space ($\vec{r}$) and momentum (represented by ν and $\vec{\Omega}$). This is clearly required since we assume in writing I as a function of $\vec{r}$, ν, and $\vec{\Omega}$ that it is sufficient to specify the phase space coordinates of the "center" of the wave packet, and that any information concerning the distribution about the center is irrelevant. Because of the uncertainty principle, which limits the wave packet spread in $\vec{r}$ and $\vec{p}$, these considerations impose a minimum on the spatial and momentum resolution possible in the equation of transfer.

3. Interference effects are ignored, since the transport equation is an equation for intensities rather than wave amplitudes. Hence, the photon density must be low; i.e., low enough so that the overlap in the tails of the wave packets is negligibly small. This restriction is somewhat too strong since, given a time resolution of interest, photons of sufficiently different frequencies do not interfere even if they coincide spatially. This fact is needed to be able to treat the source photons as incoherent.

4. Collisions occur instantaneously, and spontaneous emission occurs instantaneously. This imposes a minimum on the time resolution that a kinetic equation can supply.

5. No diffraction or reflection is possible. These phenomena depend upon interference among the waves arising from different scattering centers, which scatter the same photon. For interference of this type to occur, two conditions must be satisfied. First, the scattering centers must be correlated (as in a crystal) and secondly, the spatial extent of the wave packet must be such that several scattering centers are encompassed by a single photon.

Simplifying Assumptions

1. Polarization is neglected. Four parameters are required to specify the state of polarization of a beam of light. Further, the state of polarization and hence these four parameters change when a photon is scattered. A proper description of radiative transfer involves four coupled transport equations. The single equation we deal with can be considered as the result of averaging this set of four equations over polarization states, assuming the light to be natural (unpolarized). The fact that four parameters are needed is easily demonstrated. They are: (a) the intensity; (b) the proportion of unpolarized light and elliptically polarized light, a decomposition that is always possible and is unique; (c) the plane of polarization of the ellipse (its orientation in space); (d) the ellipticity (the ratio of the axes).

2. Refraction and dispersion is neglected. That is, the refractive index is taken as unity. If this index is not unity, photons will not move at the vacuum speed of light. In addition, if this index depends upon space, photons will not stream in straight lines between collisions but will undergo (continuous) refraction. In addition, if the index depends upon time, a photon will (continuously) change its frequency as it streams between collisions (dispersion). The origin of these effects is interference between the scattered wave (from a single scattering center) in the near forward direction and the incident wave.

3. The medium is assumed isotropic. That is, in the fluid rest frame there is no preferential direction in the matter. Hence, $\sigma_a(\nu)$, $\sigma_s(\nu)$, and $B(\nu,T)$ do <u>not</u> depend on $\vec{\Omega}$, and $\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})$ depends only upon $\vec{\Omega}'\cdot\vec{\Omega}$, <u>not</u> $\vec{\Omega}'$ and $\vec{\Omega}$ separately.

4. Moving medium effects are neglected. The fact that the fluid is moving does, to an observer at rest, introduce a

preferred direction in the matter. Then σ_a, σ_s, and B depend upon $\vec{\Omega}$ and the differential scattering cross section depends upon $\vec{\Omega}'$ and $\vec{\Omega}$ separately. These are relativistic effects, of the order of $\vec{u}/c$.

These four effects can be, and have been, incorporated into a kinetic description of radiative transfer. With the possible exception of $\vec{v}/c$ terms, they are generally unimportant.

B. An Eulerian Derivation of the Equation of Radiative Transfer

We write

$$d\vec{r} = dxdydz \quad , \tag{57}$$

$$d\vec{\Omega} = \sin\theta \, d\theta \, d\phi = d\mu d\phi \quad , \tag{58}$$

and consider a six-dimensional "cube" dV fixed in space such that the number of photons in this cube at time t is

$$\# \text{ of photons} = f(\vec{r},\nu,\vec{\Omega},t)d\vec{r}d\vec{\Omega}d\nu = fdV \quad . \tag{59}$$

The time rate of change of the number of photons in this cube is given by

$$\text{change} = \frac{\partial}{\partial t}(fdV) = (dV)\frac{\partial f}{\partial t} \quad . \tag{60}$$

We equate this change to the time rate of change of sources and sinks, namely: streaming, absorption; outscattering, inscattering, and emission. We have, for each of these terms,

$$\begin{matrix}\text{net streaming} \\ \text{out of cube}\end{matrix} = \Big[\frac{\partial}{\partial x}(\dot{x}f) + \frac{\partial}{\partial y}(\dot{y}f) + \frac{\partial}{\partial z}(\dot{z}f) + \frac{\partial}{\partial \nu}(\dot{\nu}f)$$

$$+ \frac{\partial}{\partial \mu}(\dot{\mu}f) + \frac{\partial}{\partial \phi}(\dot{\phi}f)\Big]dV \quad , \tag{61}$$

$$\text{absorption} = c\sigma_a f dV \quad , \tag{62}$$

$$\text{outscattering} = c dV \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \; \sigma_s(\nu\to\nu', \vec{\Omega}'\cdot\vec{\Omega}) f(\nu, \vec{\Omega}) \quad , \tag{63}$$

$$\text{inscattering} = c dV \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \; \sigma_s(\nu'\to\nu, \vec{\Omega}'\cdot\vec{\Omega}) f(\nu', \vec{\Omega}') \quad , \tag{64}$$

$$\text{emission} = q dV \quad . \tag{65}$$

Thus, the conservation (photon balance) equation is:

$$\frac{\partial f(\nu,\vec{\Omega})}{\partial t} + c\vec{\Omega}\cdot\vec{\nabla} f(\nu,\vec{\Omega}) = q(\nu) - c\sigma_a(\nu) f(\nu,\vec{\Omega})$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' [\sigma_s(\nu'\to\nu, \vec{\Omega}\cdot\vec{\Omega}') f(\nu', \vec{\Omega}') - \sigma_s(\nu\to\nu', \vec{\Omega}\cdot\vec{\Omega}') f(\nu, \vec{\Omega})], \tag{66}$$

where we have neglected refraction and dispersion, i.e., set

$$\dot{\nu} = \dot{\mu} = \dot{\phi} = 0 \quad , \tag{67}$$

and set

$$\dot{x} = c\Omega_x; \quad \dot{y} = c\Omega_y; \quad \dot{z} = c\Omega_z \quad . \tag{68}$$

If we rewrite Eq. (66) in terms of

$$I(\nu,\vec{\Omega}) \equiv ch\nu f(\nu,\vec{\Omega}) \quad , \tag{69}$$

defining

$$S(\nu) = h\nu q(\nu) \quad , \tag{70}$$

we have

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + \sigma_a(\nu)I(\nu,\vec{\Omega}) = S(\nu) + \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'$$

$$\left[\frac{\nu}{\nu'}\,\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})I(\nu',\vec{\Omega}') - \sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}')I(\nu,\vec{\Omega})\right] \quad . \qquad (71)$$

Using

$$\sigma_s(\nu) = \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\,\sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}') \quad , \qquad (72)$$

and

$$\sigma = \sigma_a + \sigma_s \quad , \qquad (73)$$

we can rewrite Eq. (71) as:

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + \sigma(\nu)I(\nu,\vec{\Omega}) = S(\nu)$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\,\frac{\nu}{\nu'}\,\sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}')I(\nu',\vec{\Omega}') \quad . \qquad (74)$$

C. A Lagrangian Derivation of the Equation of Radiative Transfer

As a packet of photons travels in matter, its number would be conserved except for the processes of absorption, scattering, and emission. We have just computed these three processes, and we can immediately write

$$\frac{d}{dt}\,(f dV) = q dV - c\sigma_a f dV - c\sigma_s f dV$$

$$+ c dV \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\,\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})f(\nu',\vec{\Omega}') \quad . \qquad (75)$$

Here, d/dt is the total, or Lagrangian, time derivative, taken along the natural path of the streaming packet of photons. That is, the term d(fdV)/dt means the difference between the value of fdV at s + ds and its value at s (where ds = cdt is an element of length along the photon path) divided by the transit time dt. This difference in value comes, in general, from both an explicit time dependence of fdV and an implicit time dependence through the other variables involved.

It should also be stressed that dV here is not fixed in space, but travels along with the packet of photons. Its size varies in time in just such a way that at any instant of time it encompasses the photons of interest. Hence,

$$\frac{d}{dt}(dV) \neq 0 \quad , \tag{76}$$

but the change in dV with time must be calculated.

The rule for differentiating a product gives (at this point it is convenient to use ΔV rather that dV, and let $\Delta V \to dV$ as appropriate in the manipulations)

$$\frac{d}{dt}(f\Delta V) = \frac{d}{dt}(f\ \Delta x \Delta y \Delta z \Delta \nu \Delta \mu \Delta \phi) = \Delta x \Delta y \Delta z \Delta \nu \Delta \mu \Delta \phi \frac{df}{dt}$$

$$+ f\ \Delta y \Delta z \Delta \nu \Delta \mu \Delta \phi \frac{d}{dt}(\Delta x) + \text{five similar terms} \quad , \tag{77}$$

or, introducing ΔV,

$$\frac{d}{dt}(f\Delta V) = \Delta V \frac{df}{dt}$$

$$+ f\Delta V\left[\frac{1}{\Delta x}\frac{d}{dt}(\Delta x) + \frac{1}{\Delta y}\frac{d}{dt}(\Delta y) + \frac{1}{\Delta z}\frac{d}{dt}(\Delta z) + \ldots\right] \quad . \tag{78}$$

Considering $\Delta x = x_2 - x_1$, we have

$$\frac{d}{dt}(\Delta x) = \frac{d}{dt}(x_2 - x_1) = \dot{x}\Big|_{x_1}^{x_2} = \left(\frac{\partial \dot{x}}{\partial x}\right)\Delta x \quad . \tag{79}$$

Hence, Eq. (78) becomes

$$\frac{d}{dt}(f\Delta v) = \Delta V \frac{df}{dt} + f\Delta V\left[\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + . . .\right] \quad . \tag{80}$$

Using the chain rule of differentiation on the df/dt term in Eq. (80) then yields

$$\frac{d}{dt}(f\Delta V) = \Delta V\left[\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} + . . .\right]$$

$$+ f\Delta V\left[\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + . . .\right] \quad , \tag{81}$$

or, combining the terms on the right hand side of Eq. (81),

$$\frac{d}{dt}(f\Delta V) = \Delta V\left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(\dot{x}f) + \frac{\partial}{\partial y}(\dot{y}f) + . . .\right] \quad . \tag{82}$$

Using this result in Eq. (75) and cancelling the common dV term gives

$$\frac{\partial f(\nu,\vec{\Omega})}{\partial t} + \frac{\partial(\dot{x}f)}{\partial x} + \frac{\partial(\dot{y}f)}{\partial y} + . . . = q(\nu) - c\sigma f$$

$$+ c\int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\ \sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})f(\nu',\vec{\Omega}') \quad . \tag{83}$$

Taking into account that photons stream in straight lines and introducing $I = ch\nu f$, Eq. (83) becomes

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + \sigma(\nu)I(\nu,\vec{\Omega}) = S(\nu)$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}')I(\nu',\vec{\Omega}') \quad , \qquad (84)$$

which is identical to the Eulerian derivation result [see Eq. (74)].

D. Boundary and Initial Conditions

Since the equation of transfer is a first order differential equation in space and time, we require boundary conditions in both variables.

We assume that the system of interest, which is arbitrary in composition and shape, is non-reentrant, by which we mean that any photon that escapes through the surface will not re-enter the system through another part of the surface. If the body is re-entrant, we enclose it in a non-reentrant hypothetical surface (such as a spherical shell) and consider the system to be bounded by the imposed, rather than the real, surface. The new system is then non-reentrant, but consists in part, of vacuum ($\sigma = S = 0$).

On physical grounds, we know it is sufficient to specify the specific intensity at each surface point in the incoming direction. Thus we have the boundary condition

$$I(\vec{r}_s,\nu,\vec{\Omega},t) = \Gamma(\vec{r}_s,\nu,\vec{\Omega},t), \quad \vec{n}\cdot\vec{\Omega} < 0 \quad , \qquad (85)$$

where Γ is a specified function of all arguments, $\vec{r}_s$ is a surface point, and $\vec{n}$ is an outward normal vector at this point.

A special case of Eq. (85) is the so-called vacuum or free surface boundary condition.

$$I(\vec{r}_s,\nu,\vec{\Omega},t) = 0, \quad \vec{n}\cdot\vec{\Omega} < 0 \quad , \qquad (85a)$$

which merely states that no photons enter the system through its bounding surface.

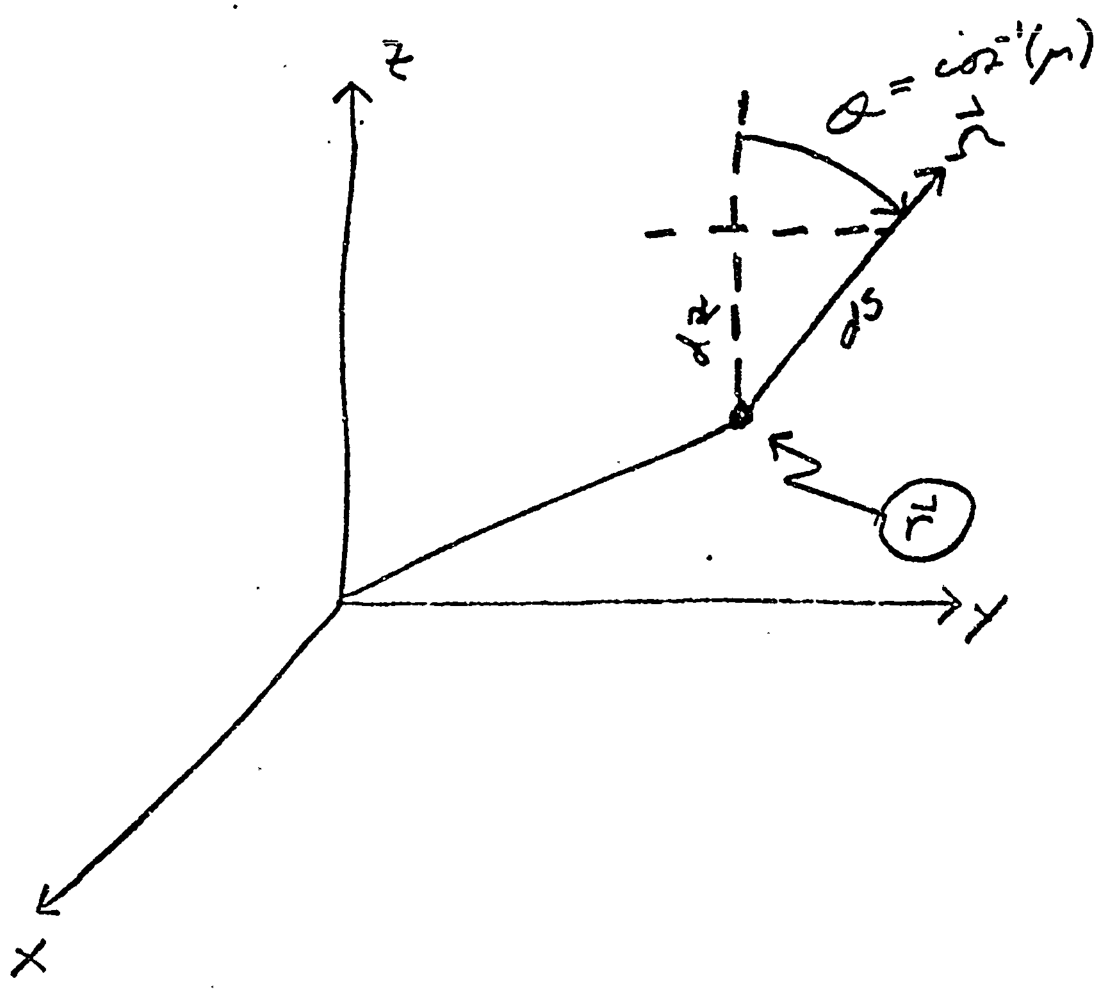
The initial condition is, with Λ a known function of all arguments,

$$I(\vec{r},\nu,\vec{\Omega},0) = \Lambda(\vec{r},\nu,\vec{\Omega}) \tag{86}$$

E. The Equation of Transfer in Various Geometries

To write the equation of transfer in any given coordinate system, we need interpret $\vec{\Omega}\cdot\vec{\nabla}I$ as a directional derivative in the $\vec{\Omega}$ direction.

Slab (Planar Geometry)



In this geometry, I is a function only of z and μ, the cosine of the angle between the z axis and $\vec{\Omega}$. We have

$$\frac{\partial I}{\partial s} = \frac{\partial I}{\partial z}\left(\frac{dz}{ds}\right) + \frac{\partial I}{\partial \mu}\left(\frac{d\mu}{ds}\right) \quad . \tag{87}$$

From the fugure we see

$$\frac{dz}{ds} = \cos\theta = \mu; \quad \frac{d\mu}{ds} = 0 \quad , \tag{88}$$

and the equation of transfer, Eq. (84), becomes

$$\frac{1}{c}\frac{\partial I(\nu,\mu)}{\partial t} + \mu\frac{\partial I(\nu,\mu)}{\partial z} + \sigma(\nu)I(\nu,\mu) = S(\nu)$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu, \vec{\Omega}\cdot\vec{\Omega}')I(\nu',\mu') \quad . \tag{89}$$

To simplify the scattering term, we expand $\sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}')$ in Legendre polynomials according to

$$\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega}) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{4\pi}\right)\sigma_{sn}(\nu'\to\nu)P_n(\vec{\Omega}\cdot\vec{\Omega}') \quad , \tag{90}$$

where

$$\sigma_{sn}(\nu'\to\nu) = 2\pi\int_{-1}^{1} d\xi\ \sigma_s(\nu'\to\nu,\xi)P_n(\xi) \quad . \tag{91}$$

We use

$$P_n(\vec{\Omega}\cdot\vec{\Omega}') = P_n(\mu)P_n(\mu')$$

$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\mu)\ P_n^m(\mu')\cos m(\phi-\phi') \quad . \tag{92}$$

Using Eq. (92) in Eq. (90), and this result in Eq. (89) gives, noting that the cosine m $(\phi - \phi')$ terms integrate to zero,

$$\frac{1}{c}\frac{\partial I(\nu,\mu)}{\partial t} + \mu \frac{\partial I(\nu,\mu)}{\partial z} + \sigma(\nu)I(\nu,\mu) = S(\nu)$$

$$+ \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right)P_n(\mu) \int_0^{\infty} d\nu' \frac{\nu}{\nu'} \sigma_{sn} (\nu'\to\nu)$$

$$\int_{-1}^{1} d\mu' \, P_n(\mu')I(\nu',\mu') \quad . \tag{93}$$

Spherically Symmetric Geometry

In this geometry, I depends upon the radial coordinate r and μ, the cosine of the angle between r and $\vec{\Omega}$. We have

$$\frac{\partial I}{\partial s} = \frac{\partial I}{\partial r}\left(\frac{dr}{ds}\right) + \left(\frac{\partial I}{\partial \mu}\right)\left(\frac{d\mu}{ds}\right) \quad , \tag{94}$$

and

$$\frac{dr}{ds} = \cos\theta = \mu \ ; \qquad \frac{d\mu}{ds} = \frac{1-\mu^2}{r} \quad . \tag{95}$$

(Note that $d\mu/ds \neq 0$ since θ is not measured with respect to an axis fixed in space).

The scattering term can be treated just as in planar geometry, and hence we have

$$\frac{1}{c}\frac{\partial I(\nu,\mu)}{\partial t} + \mu \frac{\partial I(\nu,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I(\nu,\mu)}{\partial \mu}$$

$$+ \sigma(\nu)I(\nu,\mu) = S(\nu) + \sum_{n=0}^{\infty} \left(\frac{2n+1}{2}\right) P_n(\mu)$$

$$\int_0^{\infty} d\nu' \frac{\nu'}{\nu} \sigma_{sn}(\nu'\to\nu) \int_{-1}^{1} d\mu' \, P_n(\mu')I(\nu',\mu') \quad . \tag{96}$$

Cylindrically Symmetric Geometry

In this geometry, I depends upon the single spatial coordinate r (the usual cylindrical coordinate), but two angles defining $\vec{\Omega}$ are needed. These may be taken as θ, the angle between the projection of $\vec{\Omega}$ in the x-y plane and the cylindrical coordinate r. Then

$$\frac{\partial I}{\partial s} = \frac{\partial I}{\partial r}\left(\frac{dr}{ds}\right) + \frac{\partial I}{\partial \theta}\left(\frac{d\theta}{ds}\right) + \frac{\partial I}{\partial \phi}\left(\frac{d\phi}{ds}\right) \quad . \tag{97}$$

One finds

$$\frac{dr}{ds} = \sin\theta \cos\phi \ ; \quad \frac{d\phi}{ds} = -\frac{1}{r}\sin\theta \sin\phi \ ; \quad \frac{d\theta}{ds} = 0 \quad , \tag{98}$$

and hence the transport equation is

$$\frac{1}{c}\frac{\partial I(\nu,\theta,\phi)}{\partial t} + \sin\theta\left[\cos\phi \frac{\partial I}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial I}{\partial \phi}\right] + \sigma(\nu)I(\nu,\theta,\phi)$$

$$= S(\nu) + \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu, \vec{\Omega}\cdot\vec{\Omega}')I(\nu',\theta',\phi') \ . \tag{99}$$

In this case no simplification in the scattering term is possible since the angular dependence of I is as general as in the original equation of transfer (i.e., two angles are required).

3-D Geometries

In general 3-D geometry, we have:

Cartesian

$$\frac{\partial I}{\partial s} = \Omega_x \frac{\partial I}{\partial x} + \Omega_y \frac{\partial I}{\partial y} + \Omega_z \frac{\partial I}{\partial z} \quad , \tag{100}$$

with

$$\Omega_x^2 + \Omega_y^2 + \Omega_z^2 = 1 \quad , \tag{101}$$

Spherical

$$\frac{\partial I}{\partial s} = \mu \frac{\partial I}{\partial r} + \frac{\eta}{r}\frac{\partial I}{\partial \Theta} + \frac{\xi}{r\sin\Theta}\frac{\partial I}{\partial \Phi} + \frac{(1-\mu^2)}{r}\frac{\partial I}{\partial \mu} - \frac{\xi\cot\Theta}{r}\frac{\partial I}{\partial \phi} \quad , \tag{102}$$

where r, Θ, Φ are the usual spherical spatial coordinates,

$$\mu = \cos\theta; \quad \eta = \sin\theta\cos\phi; \quad \xi = \sin\theta\sin\phi \quad , \tag{103}$$

with

$$\mu^2 + \eta^2 + \xi^2 = 1 \quad . \tag{104}$$

Here θ is the polar angle betwen r and $\vec{\Omega}$, and ϕ is the azimuthal angle between the projection of $\vec{\Omega}$ in the plane perpendicular to r and any reference axis in this plane.

Cylindrical

$$\frac{\partial I}{\partial s} = \mu \frac{\partial I}{\partial r} + \frac{\eta}{r}\frac{\partial I}{\partial \Theta} + \xi \frac{\partial I}{\partial z} - \frac{\eta}{r}\frac{\partial I}{\partial \phi} \quad , \tag{105}$$

where r, Θ, and z are the usual cylindrical spatial coordinates,

$$\xi = \cos\theta; \quad \mu = \sin\theta\cos\phi; \quad \eta = \sin\theta\sin\phi \quad , \tag{106}$$

with

$$\xi^2 + \mu^2 + \eta^2 = 1 \quad . \tag{107}$$

Here θ is the polar angle between the axis and $\vec{\Omega}$, and ϕ is an azimuthal angle between r and the projection of $\vec{\Omega}$ in the x-y plane.

F. The Integral Form of the Equation of Transfer

We first consider the time independent equation of transfer. We have, from Eq. (84),

$$\vec{\Omega}\cdot\vec{\nabla}I(\vec{r},\nu,\vec{\Omega}) + \sigma(\vec{r},\nu)I(\vec{r},\nu,\vec{\Omega}) = Q(\vec{r},\nu,\vec{\Omega}) \quad , \tag{108}$$

where $Q(\vec{r},\nu,\vec{\Omega})$ is the total (emission + scattering) source

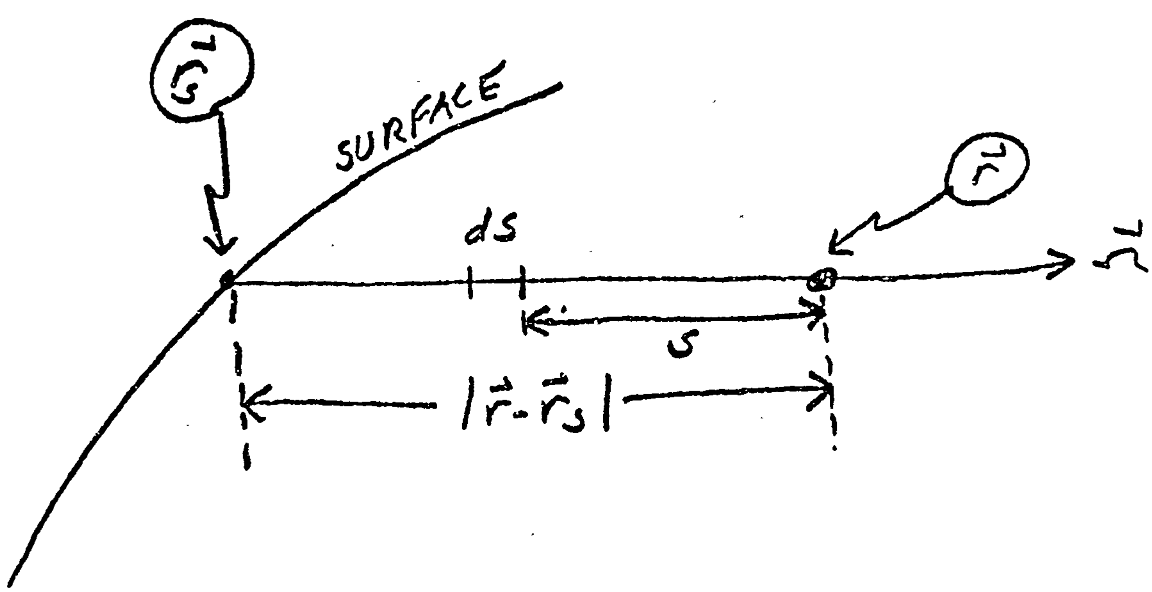
$$Q(\vec{r},\nu,\vec{\Omega}) = S(\vec{r},\nu)$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \, \frac{\nu}{\nu'} \, \sigma_s(\vec{r},\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}')I(\vec{r},\nu',\vec{\Omega}') \quad . \tag{109}$$

The boundary condition on Eq. (108) is

$$I(\vec{r}_s,\nu,\vec{\Omega}) = \Gamma(\vec{r}_s,\nu,\vec{\Omega}) \quad , \quad \vec{n}\cdot\vec{\Omega} < 0 \quad . \tag{110}$$

We interpret $\vec{\Omega}\cdot\vec{\nabla}$ as a directional derivative in the direction $\vec{\Omega}$, and introduce the distance s from the point $\vec{r}$ in the $-\vec{\Omega}$ direction.



We write Eq. (108) at the point $\vec{r} - s\vec{\Omega}$ as, suppressing the frequency variable

$$-\frac{\partial I(\vec{r} - s\vec{\Omega},\vec{\Omega})}{\partial s} + \sigma(\vec{r} - s\vec{\Omega})I(\vec{r} - s\vec{\Omega},\vec{\Omega}) = Q(\vec{r} - s\vec{\Omega},\vec{\Omega}) \quad . \tag{111}$$

This is a first order equation that can be integrated by introducing an integrating factor. The result is:

$$I(\vec{r} - s\vec{\Omega},\vec{\Omega}) = I(\vec{r} - s_o\vec{\Omega})\exp[\int_{s_o}^{s} ds''\sigma(\vec{r} - s''\vec{\Omega})]$$

$$+ \int_{s}^{s_o} ds'Q(\vec{r} - s'\vec{\Omega},\vec{\Omega})\exp[\int_{s'}^{s} ds''\sigma(\vec{r} - s''\vec{\Omega})] \quad , \tag{112}$$

where s_o is an arbitrary point along s. Setting s = 0 in Eq. (112) gives

$$I(\vec{r},\vec{\Omega}) = I(\vec{r} - s_o\vec{\Omega})\exp[-\int_{0}^{s_o} ds''\sigma(\vec{r} - s''\vec{\Omega})]$$

$$+ \int_{0}^{s_o} ds'Q(\vec{r} - s'\vec{\Omega},\vec{\Omega})\exp[-\int_{0}^{s'} ds''\sigma(\vec{r} - s''\vec{\Omega})] \quad . \tag{113}$$

To apply the boundary condition, we choose s_o much that

$$\vec{r} - s_o\vec{\Omega} = \vec{r}_s \Rightarrow s_o = |\vec{r} - \vec{r}_s| \quad . \tag{114}$$

Then

$$I(\vec{r} - s_o\vec{\Omega},\vec{\Omega})\Big|_{s_o=|\vec{r}-\vec{r}_s|} = \Gamma(\vec{r}_s,\vec{\Omega}) \quad . \tag{115}$$

Then, setting $s_o = |\vec{r} - \vec{r}_s|$ in Eq. (113) we obtain

$$I(\vec{r},\vec{\Omega}) = \Gamma(\vec{r}_s,\vec{\Omega})\exp[-\int_0^{|\vec{r}-\vec{r}_s|} ds''\ \sigma(\vec{r} - s''\vec{\Omega})]$$

$$+ \int_0^{|\vec{r}-\vec{r}_s|} ds' Q(\vec{r} - s'\vec{\Omega},\vec{\Omega})\exp[-\int_0^{s'} ds''\sigma(\vec{r} - s''\vec{\Omega})] \quad , \qquad (116)$$

which is the integral form of the equation of transfer. We see according to Eq. (116) that $I(\vec{r},\vec{\Omega})$ is the sum of two terms: (1) the intensity incident upon the surface exponentially attenuated by collisions along the path; and (2) a contribution due to emission and scattering into the beam from each path length element ds' along $\vec{\Omega}$, also exponentially attenuated.

The quantity

$$\tau(\vec{r},\vec{r}_s) \equiv \int_0^{|\vec{r}-\vec{r}_s|} ds''\sigma(\vec{r} - s''\vec{\Omega}) \quad , \qquad (117)$$

which occurs in Eq. (116) is referred to as the <u>optical depth</u> or <u>optical path length</u> between the points $\vec{r}$ and $\vec{r}_s$. It is clear from Eq. (116) that it is the optical depth between two points that is the relevant quantity in calculating the exponential attenuation of a beam of photons in traveling from one point to another.

One could repeat this derivation in the time dependent case. Omitting the details, the result is:

$$I(\vec{r},\vec{\Omega},t) = [\int_0^{|\vec{r}-\vec{r}_s|} ds' Q(\vec{r} - s'\vec{\Omega},\vec{\Omega},t - s'/c) \ \cdot$$

$$\cdot \exp[-\int_0^{s'} ds''\sigma(\vec{r} - s''\vec{\Omega},t - s''/c)] +$$

$$+ \Gamma(\vec{r}_s,\vec{\Omega},t - |\vec{r} - \vec{r}_s|/c)H(ct - |\vec{r} - \vec{r}_s|)$$

$$\cdot\exp[-\int_0^{|\vec{r}-\vec{r}_s|} ds''\sigma(\vec{r} - s''\vec{\Omega},t - s''/c)]$$

$$+ \Lambda(\vec{r} - ct\vec{\Omega},\vec{\Omega})H(|\vec{r} - \vec{r}_s| - ct)$$

$$\cdot\exp[-\int_0^{ct} ds''\sigma(\vec{r} - s''\vec{\Omega},t - s''/c)] \quad , \tag{118}$$

where H(z) is the Heaviside function

$$H(z) = \begin{cases} 0 & z < 0 \\ 1 & z > 0 \end{cases} .$$

This equation, the so-called formal solution of the equation of transfer, is algebraically quite complex, but the physical interpretation is simple; namely, photons of direction $\vec{\Omega}$, which are at a point $\vec{r}$ at time t must have originated at some point $\vec{r} - s\vec{\Omega}$ at time $t - s/c$, due to the finite speed of light. One must also account for the exponential attenuation, based upon the optical depth between $\vec{r}$ and $\vec{r} - s\vec{\Omega}$. This is the entire content of Eq. (118).

G. Peierl's Equation

We consider the special case of time independent radiative transfer with isotropic scattering and no incident photons ($\Gamma = 0$). In this case Eq. (116) becomes

$$I(\vec{r},\vec{\Omega}) = \int_0^\infty ds'Q(\vec{r} - s'\vec{\Omega}) \exp[-\int_0^{s'} ds''\sigma(\vec{r} - s''\vec{\Omega})] \quad . \tag{119}$$

In writing Eq. (119), we have used the fact that $Q(\vec{r},\vec{\Omega})$ is, in fact, independent of $\vec{\Omega}$ because the emission and scattering (by assumption) are isotropic. We have further taken the upper limit of integration as ∞ since Q is zero for $s' > |\vec{r} - \vec{r}_s|$.

Integrating Eq. (119) over all $\vec{\Omega}$, recalling that the radiative energy density is given by

$$E(\vec{r}) = \frac{1}{c} \int_{4\pi} d\vec{\Omega} I(\vec{r},\vec{\Omega}) \quad , \tag{120}$$

[the function E in Eq. (120) is really the energy density per unit frequency since no integral over ν is involved], we obtain the result

$$cE(\vec{r}) = \int_{4\pi} d\vec{\Omega} \int_0^\infty ds' Q(\vec{r} - \vec{s}'\vec{\Omega}) \exp\left[- \int_0^{s'} ds'' \sigma(\vec{r} - s''\vec{\Omega})\right] . \tag{121}$$

We define

$$\vec{r}' = \vec{r} - s'\vec{\Omega} \quad , \tag{122}$$

and hence

$$s' = |\vec{r} - \vec{r}'| \quad . \tag{123}$$

Then Eq. (121) becomes

$$cE(\vec{r}) = \int_{4\pi} d\vec{\Omega} \int_0^\infty d|\vec{r} - \vec{r}'| Q(\vec{r}') \quad .$$

$$\cdot \exp\left[- \int_0^{|\vec{r}-\vec{r}'|} ds'' \sigma(\vec{r} - s''\vec{\Omega})\right] \quad . \tag{124}$$

Recognizing the exponent in Eq. (124) as the optical depth $\tau(\vec{r},\vec{r}')$ and grouping terms, we have

$$cE(\vec{r}) = \int_0^\infty d|\vec{r} - \vec{r}'|\,|\vec{r} - \vec{r}'|^2 \int_{4\pi} d\vec{\Omega}\left[\frac{Q(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\, e^{-\tau(\vec{r},\vec{r}')}\right]. \quad (125)$$

We now recognize that $|\vec{r} - \vec{r}'|^2 d|\vec{r} - \vec{r}'|d\vec{\Omega}$ is just a volume element in spherical coordinates, centered around the point $\vec{r}$. One can rewrite this volume element as simply $d\vec{r}'$, without reference to any particular coordinate system. Hence Eq. (125) can be rewritten

$$cE(\vec{r}) = \int_V d\vec{r}'\, \frac{Q(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\, e^{-\tau(\vec{r},\vec{r}')} \quad , \qquad (126)$$

where the integration extends over the volume of the system.

The function $Q(\vec{r})$ in Eq. (126) is given by

$$Q(\vec{r}) = Q(\vec{r},\nu) = S(\vec{r},\nu) + \int_0^\infty d\nu'\, \frac{\nu}{\nu'}\, \sigma_s(\vec{r},\nu'\rightarrow\nu)cE(\vec{r},\nu') \quad ,(127)$$

and hence Eq. (126) becomes

$$cE(\vec{r},\nu) =$$

$$\int_V d\vec{r}'\, \frac{4\pi\left[S(\vec{r}',\nu) + \int_0^\infty d\nu'\, \frac{\nu}{\nu'}\, \sigma_s(\vec{r}',\nu'\rightarrow\nu)cE(\vec{r}',\nu')\right]e^{-\tau(\vec{r},\vec{r}')}}{4\pi|\vec{r}-\vec{r}'|^2} . \quad (128)$$

Equation (128) is an integral equation for $E(\vec{r},\nu)$ and is known as Peierls' equation. We have introduced a factor of 4π in both the numerator and denominator of Eq. (128) to aid in the physical interpretation of this result. The term $4\pi\, Q(\vec{r}')$ is just the angle integrated total (emission plus scattering) source at a point $\vec{r}'$. To obtain the contribution of this source to $cE(\vec{r})$, one must attenuate it by the proper exponential, namely $e^{-\tau(\vec{r},\vec{r}')}$ which is the noncollision probability. One must also introduce the geometric attenuation due to spherical divergence, namely the area of the spherical shell at point $\vec{r}$ with center $\vec{r}'$. This is just $4\pi|\vec{r} - \vec{r}'|^2$, the denominator in Eq. (128).

If one considers Peierls' equation in the standard three one-dimensional geometries, one finds from Eq. (126):

Planar Geometry

$$cE(z) = 2\pi \int_0^R dz' \, E_1(|\tau - \tau'|)Q(z') \tag{129}$$

where

$$E_n(z) \equiv \int_1^\infty dt \, \frac{e^{-zt}}{t^n} = \int_0^1 dt \, t^{n-2} \, e^{-z/t} \tag{130}$$

is the standard n^{th} order exponential integral, and

$$\tau(z) \equiv \int_0^z dz''\sigma(z'') \quad . \tag{131}$$

The slab here extends over the range $0 \leq z \leq R$, and $\sigma = \sigma(z)$, an arbitrary function of space.

For spherical and cylindrical systems, one obtains relatively simple results only for homogeneous systems (a cross section σ independent of space). These results are:

Spherical

$$cE(r) = 2\pi \int_0^R dr' \, \frac{r'}{r} \, [E_1(\sigma|r-r'|) - E_1(\sigma|r+r'|)]Q(r') \quad . \tag{132}$$

Cylindrical

$$cE(r) = 4\pi\sigma \int_0^R dr'r'Q(r') \int_1^\infty dyI_o(\sigma r_< y)K_o(\sigma r_> y) \quad . \tag{133}$$

where $I_0(z)$ and $K_0(z)$ are the usual Bessel functions, and

$$r_< = \min (r,r') \quad ,$$

$$r_> = \max (r,r') \quad . \tag{134}$$

H. Induced Processes and Local Thermodynamic Equilibrium

The equation of transfer considered thus far may properly be termed the classical equation of transfer since its derivation was based solely on classical physics concepts. We now modify this equation to account for so-called induced processes, a non-classical concept.

Specifically, we consider the manifestation in the equation of transfer of the quantum statistics obeyed by photons. Since photons are bosons, both the processes of emission and scattering are enhanced by the number of photons already in the final state following the interaction. This enhancement is generally referred to as resulting from induced processes. The quantititive statement of this enhancement is simply stated as: If P represents the basic rate of a photon event (emission or scattering) then, due to induced effects, the actual rate P' is given by

$$P' = P(1 + n) \quad , \tag{135}$$

where n is the number of photons in the final state of the transition. In the present context, the final state corresponds to the basic, or unit cell, of phase space. In terms of the distribution function $f(\vec{r},\nu,\vec{\Omega},t)$ introduced earlier, the number of photons at time t in a unit cell of $\vec{r},\nu,\vec{\Omega}$ space is given by

$$n = \int d\vec{r} \int_{\Delta} d\nu \int d\vec{\Omega}\, f(\vec{r},\nu,\vec{\Omega},t) \quad , \tag{136}$$

where Δ denotes the unit cell of phase space. It is convenient to transform from ν, $\vec{\Omega}$ space to $\vec{p}$ (momentum) space. We have, representing the momentum differential in spherical coordinates

$$d\vec{p} = p^2 dp d\vec{\Omega} \quad , \tag{137}$$

or, since $p = h\nu/c$ for photons,

$$d\vec{p} = (h/c)^3 \nu^2 d\nu d\vec{\Omega} \quad . \tag{138}$$

Introducing the specific intensity $I = ch\nu f$ in Eq. (136) and making use of Eq. (138), we obtain

$$n = \frac{c^2}{h^4} \int d\vec{r} \int_{\Delta} d\vec{p} \; I(\vec{r},\nu,\vec{\Omega},t)/\nu^3 \tag{139}$$

where Δ now denotes the basic cell in $\vec{r}$, $\vec{p}$ space. This basic element is

$$\Delta = \Delta\vec{p}\Delta\vec{r} = h^3/2 \quad , \tag{140}$$

with the factor of 2 arising because each h^3 of phase space can accommodate two photons, one of each polarization state. Hence Eq. (139) yields

$$n = \frac{c^2}{h^4 \nu^3} I\Delta = \frac{c^2}{2h\nu^3} I \quad , \tag{141}$$

and thus

$$P' = P\left[1 + c^2 I/2h\nu^3\right] \quad . \tag{142}$$

Equation (142) implies that the rates of emission and scattering in the equation of transfer should be increased by a factor $1 + c^2 I/2h\nu^3$, where the frequency and angle arguments of the specific intensity correspond to the state of the photon after the emission or scattering process has occurred. With this change, the classical equation of transfer, Eq. (71), becomes

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} I(\nu,\vec{\Omega}) = S(\nu)\left[1 + \frac{c^2 I(\nu,\vec{\Omega})}{2h\nu^3}\right] - \sigma_a(\nu) I(\nu,\vec{\Omega})$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}') I(\nu',\vec{\Omega}')\left[1 + \frac{c^2 I(\nu,\vec{\Omega})}{2h\nu^3}\right]$$

$$- \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}') I(\nu,\vec{\Omega})\left[1 + \frac{c^2 I(\nu',\vec{\Omega}')}{2h\nu'^3}\right] \quad . \quad (143)$$

Equation (143) is the equation of radiative transfer including the effects of induced processes. It can be seen that induced scattering severely complicates the equation of transfer in that it leads to nonlinear terms, quadratic in the intensity. It should be noted that if the scattering is coherent, i.e.,

$$\sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}') = \sigma_s(\vec{\Omega}\cdot\vec{\Omega}')\delta(\nu - \nu') \quad , \quad (144)$$

then the induced inscattering and outscattering terms identically cancel one another, and the equation of transfer again becomes linear. The induced contribution to emission remains, however, but this term is always linear in character. As will be discussed later, these induced scattering terms are necessary in the equation of transfer for the scattering operator to give the correct equilibrium distribution, namely a Planck function. The neglect of induced processes leads to the Wien, rather than the Planck, function as the equilibrium distribution for the specific intensity. A final note of interest concerning induced processes is that they result from a physical principle closely connected with the Pauli exclusion principle. The Pauli principle reduces the probability of fermion events by the factor $1 - n$, and hence if the specific intensity described fermions the appropriate factors in Eq. (143) would be $1 - c^2I/2h\nu^3$.

Another item of interest to consider here is the concept of <u>local thermodynamic equilibrium</u> (LTE). With reference to Eq. (143), the source term S represents the source of photons due to spontaneous emission from atoms, and the cross sections σ_a and σ_s determine the interaction of photons with the matter. In general, these three quantities depend upon the <u>microscopic</u> description of the atoms that compose the matter, i.e., the population of the various states of the atoms, and there is no simple relationship between the three quantities. A simplifying assumption in this regard often invoked in radiation-hydrodynamic work is the LTE assumption. It is assumed that the properties of the matter are dominated by atomic collisions, which establish thermodynamic equilibrium at position $\vec{r}$ and time t, and that the radiation field, even if it deviates substantially from the equilibrium Planck distribution, does not affect this equilibrium. That is, at a given instant of time and point in space it suffices to specify, in addition to the atomic composition, two thermodynamic quantities much as temperature and density in order to compute the source term S, absorption coefficient σ_a, and scattering coefficient σ_s. Equilibrium statistical mechanics, together with quantum mechanics, can then in principle be used to compute S, σ_a, and σ_s. In particular, the Saha and Boltzmann laws, appropriate to thermodynamic equilibrium, can be used to determine the relative abundance of the ionic species and the population of the states within a given ionic species. The LTE assumption also leads to a simple relationship between S and σ_a, as we now show.

As it stands, Eq. (143) is not restricted to LTE situations, but describes a more general class of problems. To see the effect of the LTE assumption on the equation of transfer, it is convenient to eliminate S and σ_a in Eq. (143) in favor of B and σ_a', defined by the relationships

$$S = \sigma_a' B \quad , \tag{145}$$

$$\sigma_a = \sigma_a'(1 + c^2 B/2h\nu^3) \quad . \tag{146}$$

At this point, B is <u>not</u> to be interpreted as the Planck functon, but is merely a new variable defined in terms of σ_a and S according to Eqs. (145) and (146). In terms of B and σ_a', Eq. (143) is written

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I = \sigma_a'(\nu)[B(\nu) - I(\nu,\vec{\Omega})]$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}')I(\nu',\vec{\Omega}')[1 + \frac{c^2 I(\nu,\vec{\Omega})}{2h\nu^3}]$$

$$- \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}')I(\nu,\vec{\Omega})[1 + \frac{c^2 I(\nu',\vec{\Omega}')}{2h\nu'^3}] \quad . \tag{147}$$

For simplicity, we take $\sigma_s = 0$ in Eq. (147), although the argument we are about to make can also be made with scattering included, using the detailed balance relationship to be discussed later.

Now, in complete thermodynamic equilibrium, the radiation field is independent of space and time and hence, in this situation with the neglect of scattering, Eq. (147) reads

$$\sigma_a'(\nu)[B(\nu) - I(\nu,\vec{\Omega})] = 0 \quad . \tag{148}$$

It is well known that in complete thermodynamic equilibrium, the equation of transfer must give the Planck black body distribution for the specific intensity I. For this to be the case, it is clear from Eq. (148) that B must be the Planck function as well. Since the LTE assumption states that the radiation field does not affect the properties of the matter, in particular the source function B, we conclude that under the LTE simplification, B is the Planck function no matter what the radiation field is. That is, under LTE we have

$$B(\nu) = \frac{2h\nu^3}{c^2} (e^{h\nu/kT} - 1)^{-1} , \tag{149}$$

where T is the local temperature of the matter. Further, use of Eq. (149) in Eq. (146) gives

$$\sigma_a'(\nu) = \sigma_a(\nu)(1 - e^{-h\nu/kT}) . \tag{150}$$

Here $\sigma_a(\nu)$ is the absorption coefficient appropriate to thermodynamic equilibrium and the exponential factor is the effective decrease in the absorption coefficient due to stimulated emission.

The form of Eq. (147), involving emission and absorption in the form $\sigma_a'(B - I)$, is the conventional way of writing the transport equation in radiative transfer, even if the LTE assumption is not invoked (in which case B is not the Planck function). However, the LTE assumption is generally made in radiation hydrodynamic work because of the vast simplification it introduces; namely thermodynamics can be used to describe the matter. In the absence of the LTE assumption, rate equations involving radiative and collisional transitions for the various ionic species and related energy levels for the atom must be solved simultaneously with the equation of transfer.

I. Black Body - Emissivity

We introduce the concept of a black body and the emissivity of a grey (non-black) body. We assume the scattering is coherent, in which case the quadratic induced scattering terms drop out, and we have as the equation of transfer in the steady state limit

$$\vec{\Omega}\cdot\vec{\nabla}I(\vec{r},\nu,\vec{\Omega}) + \sigma_s(\vec{r},\nu)I(\vec{r},\nu,\vec{\Omega}) = \sigma_a'(\vec{r},\nu)\left[B(\nu,T) - I(\vec{r},\nu,\vec{\Omega})\right]$$

$$+ \int_{4\pi} d\vec{\Omega}'\ \sigma_s(\vec{r},\nu,\vec{\Omega}\cdot\vec{\Omega}')I(\vec{r},\nu,\vec{\Omega}') \quad . \tag{151}$$

We envision a convex body of arbitrary shape, whose characteristic size L and characteristic radius of curvature R is large compared to $1/\sigma_a'$. Then locally, near any surface point, this body can be treated as a semi-infinite halfspace, and the equation of transfer becomes

$$\mu\,\frac{\partial I(z,\nu,\mu)}{\partial z} + \sigma_s(z,\nu)I(z,\nu,\mu) = \sigma_a'(z,\nu)\left[B(\nu,T) - I(z,\nu,\mu)\right]$$

$$+ \int_{4\pi} d\vec{\Omega}'\ \sigma_s(z,\nu,\vec{\Omega}\cdot\vec{\Omega}')I(z,\nu,\mu') \quad , \tag{152}$$

where z is a coordinate perpendicular to the surface.

A <u>black body</u> is a large ($L\sigma_a' >> 1$, $R\sigma_a' >> 1$) purely absorbing system, i.e., $\sigma_s = 0$.

We compute the radiative flux leaving the surface of a black body with a constant temperature. The equation to be solved is Eq. (152) with $\sigma_s = 0$ and B constant in space, i.e.,

$$\mu\,\frac{\partial I(\tau,\mu)}{\partial \tau} = B - I(\tau,\mu) \ , \quad 0 \le \tau < \infty \quad , \tag{153}$$

with boundary conditions

$$I(0,\mu) = 0 \qquad \mu > 0 \quad , \tag{154}$$

$$I(\infty,\mu) < \infty \quad . \tag{155}$$

Here we have suppressed the frequency variable and introduced the optical depth

$$\tau = \int_0^z dz' \; \sigma_a'(z') \quad . \tag{156}$$

The solution to this problem is

$$I(\tau,\mu) = B(1 - e^{-\tau/\mu}) \; , \quad \mu > 0 \; ,$$
$$I(\tau,\mu) = B \qquad \mu < 0 \; . \tag{157}$$

The flux leaving the surface is defined as

$$F_{out} = 2\pi \int_{-1}^{0} d\mu |\mu| \int_0^\infty d\nu \; I(0,\mu) \quad , \tag{158}$$

and we find

$$F_{out} = \sigma T^4 \quad , \tag{159}$$

where σ is the Stefan-Boltzman constant.

If this halfspace had a scattering component ($\sigma_s \neq 0$), the outscattering flux would be smaller (we prove this shortly) and ε, the emissivity, defined as

$$\varepsilon \equiv \frac{F_{out}}{\sigma T^4} \quad , \tag{160}$$

is less than unity (and obviously greater than zero).

We now define, and prove, Kirchoff's law. We consider two halfspace problems, each with the same absorption and scattering cross sections.

<u>Problem #1</u> - A constant temperature halfspace with a vacuum boundary condition. The transport problem is then

$$\mu \frac{\partial I_1}{\partial z} + \sigma_s I_1 = \sigma_a'(B - I_1) + \int_{4\pi} d\vec{\Omega}' \sigma_s(\vec{\Omega}\cdot\vec{\Omega}') I_1(\mu') \quad , \tag{161}$$

with boundary conditions

$$I_1(0,\nu,\mu) = 0 , \quad \mu > 0 ,$$

$$I_1(\infty,\nu,\mu) < \infty \quad \mu < 0 . \tag{162}$$

The emissivity is given by

$$\varepsilon = \frac{2\pi \int_{-1}^{0} d\mu|\mu| \int_{0}^{\infty} d\nu I_1(0,\nu,\mu)}{\sigma T^4} . \tag{163}$$

Problem #2 - A zero temperature halfspace with a Planckian boundary condition. This transport problem is

$$\mu \frac{\partial I_2}{\partial z} + \sigma_s I_2 = - \sigma_a' I_2 + \int_{4\pi} d\vec{\Omega}' \sigma_s(\vec{\Omega}\cdot\vec{\Omega}') I_2(\mu') , \tag{164}$$

with boundary conditions

$$I_2(0,\nu,\mu) = B \quad \mu > 0 ,$$

$$I_2(\infty,\nu,\mu) < \infty \quad \mu < 0 , \tag{165}$$

and the probability of absorption is

$$p = \frac{F_{in} - F_{out}}{F_{in}} = 1 - \frac{F_{out}}{F_{in}} , \tag{166}$$

or

$$p = 1 - \frac{2\pi \int_{-1}^{0} d\mu|\mu| \int_{0}^{\infty} d\nu I_2(0,\nu,\mu)}{2\pi \int_{0}^{1} d\mu\mu \int_{0}^{\infty} d\nu B} , \tag{167}$$

or

$$p = 1 - \frac{2\pi \int_{-1}^{0} d\mu|\mu| \int_{0}^{\infty} d\nu I_2(0,\nu,\mu)}{\sigma T^4} \quad . \tag{168}$$

A comparison of these two problems shows that

$$I_2(z,\nu,\mu) = B - I_1(z,\nu,\mu) \quad , \tag{169}$$

and using this in Eq. (168) shows

$$\varepsilon = p \quad ; \tag{170}$$

This is Kirchoff's Law. The emissivity of a non-black body is equal to the probability of absorption of an incident Planck distribution. Since it is clear that $0 \leq p \leq 1$, it then follows that

$$0 \leq \varepsilon \leq 1 \quad . \tag{171}$$

That is, no body at a constant temperature can radiate more than a black body.

We can also define frequency dependent emissivities ε_ν and absorption probabilities p_ν as

$$\varepsilon_\nu = \frac{2\pi \int_{-1}^{0} d\mu|\mu| I_1(0,\nu,\mu)}{2\pi \int_{-1}^{0} d\mu|\mu| B} = \frac{2}{B} \int_{-1}^{0} d\mu|\mu| I_1(0,\nu,\mu) \tag{172}$$

and

$$p_\nu = 1 - \frac{2}{B} \int_{-1}^{0} d\mu|\mu| I_2(0,\nu,\mu) \quad . \tag{173}$$

The same analysis just performed immediately shows that

$$\varepsilon_\nu = p_\nu \quad , \tag{174}$$

and hence

$$0 \leq \varepsilon_\nu \leq 1 \quad . \tag{175}$$

J. Transport (Steady State) in a Vacuum

In this case, the transport equation is simply

$$\vec{\Omega}\cdot\vec{\nabla}I(\vec{r},\nu,\vec{\Omega}) = 0 \quad , \tag{176}$$

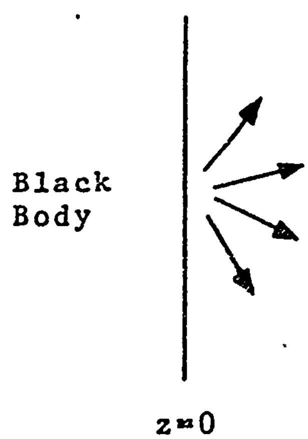
or

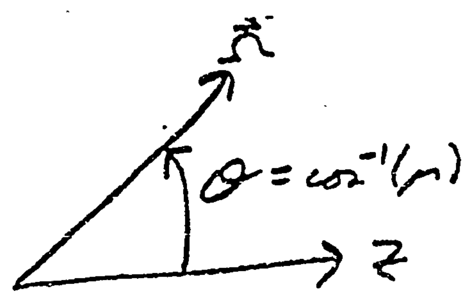
$$\frac{\partial I}{\partial s} = 0 \Rightarrow I = \text{constant} \quad . \tag{177}$$

That is, in a vacuum the specific intensity of radiation along any ray is a constant.

Lambert's Law

Consider black body radiation from a surface:





Define the directional flux at z = 0 as

$$F(0,\mu) \equiv \int_0^\infty d\nu\mu I(0,\nu,\mu) \quad . \tag{178}$$

But $I(0,\nu,\mu)$ is just B [see Eq. (157)]. Hence

$$F(0,\mu) = \int_0^\infty d\nu\mu B = \frac{\sigma T^4}{\pi}\cos\theta \quad . \tag{179}$$

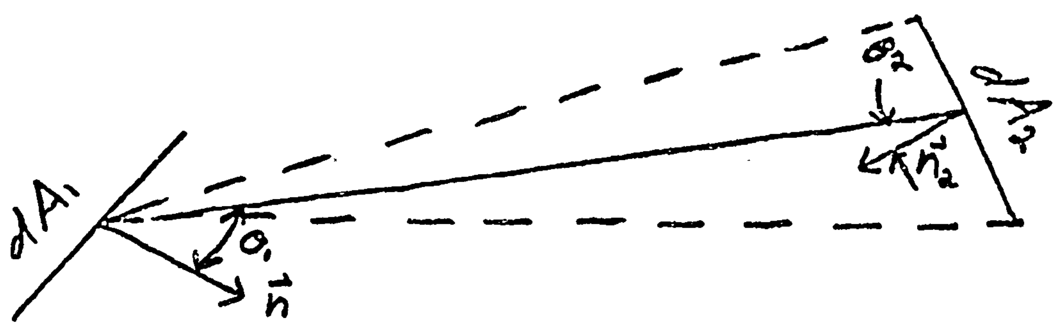
This is a Lambert's Law. The total flux radiated by the black body is then

$$F(0) = 2\pi \int_0^1 d\mu F(0,\mu) = \sigma T^4 \quad , \tag{180}$$

a result we've seen before.

View Factors

Consider two differential areas dA_1 and dA_2 separated by a distance r_{12}, each radiating as a black body.



In this picture $\vec{n}$ is a vector normal to dA, and θ is the angle between $\vec{n}$ and the line separating dA_1 and dA_2. Let $dQ_{\overrightarrow{12}}$ be the radiative energy, per unit time, leaving dA_1, which strikes dA_2. We have, by Lambert's law,

$$dQ_{\overrightarrow{12}} = \left(\frac{\sigma T_1^4}{\pi}\cos\theta_1\right)dA_1 d\vec{\Omega} \quad , \tag{181}$$

where $d\vec{\Omega}$ is the solid angle represented by the dotted lines in the figure. Using

$$d\vec{\Omega} = \frac{(dA_2)_\perp}{r_{12}^2} , \tag{182}$$

where $(dA_2)_\perp$ is the projection of dA_2 on a plane perpendicular to the line connecting dA_1 and dA_2, we have

$$dQ_{\overrightarrow{12}} = \left(\frac{\sigma T^4}{\pi} \cos\theta_1\right) dA_1 \frac{(dA_2)_\perp}{r_{12}^2} , \tag{183}$$

But

$$(dA_2)_\perp = \cos\theta_2 dA_2 . \tag{184}$$

Thus we have

$$dQ_{\overrightarrow{21}} = \left(\frac{\sigma T_1^4}{\pi}\right) \left(\frac{\cos\theta_1 \cos\theta_2}{r_{12}^2}\right) dA_1 dA_2 . \tag{185}$$

Similarly, the radiation from dA_2 which strikes dA_1 is

$$dQ_{\overrightarrow{21}} = \left(\frac{\sigma T_2^4}{\pi}\right) \left(\frac{\cos\theta_1 \cos\theta_2}{r_{12}^2}\right) dA_1 dA_2 . \tag{186}$$

The net flux from dA_1 to dA_2 is then

$$dQ_{12} \equiv dQ_{\overrightarrow{12}} - dQ_{\overrightarrow{21}} = \frac{\sigma}{\pi} \left(T_1^4 - T_2^4\right) \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1 dA_2 . \tag{187}$$

If one considers radiative transfer between two bodies of finite area, say A_1 and A_2, we then have for the net energy transferred,

$$Q_{12} \equiv \frac{\sigma}{\pi} (T_1^4 - T_2^4) \int_{A_1} dA_1 \int_{A_2} dA_2 \left(\frac{\cos\theta_1 \cos\theta_2}{r_{12}^2}\right) , \tag{188}$$

where the integration is over all dA_1 and dA_2, which "see" each other.

This result is conventionally written

$$Q_{12} = A_1 F_{12} \, \sigma(T_1^4 - T_2^4) , \tag{189}$$

or

$$Q_{12} = A_2 F_{21} \, \sigma(T_1^4 - T_2^4) , \tag{190}$$

where

$$F_{12} \equiv \frac{1}{\pi A_1} \int_{A_1} dA_1 \int_{A_2} dA_2 \left(\frac{\cos\theta_1 \cos\theta_2}{r_{12}^2}\right) , \tag{191}$$

$$F_{21} \equiv \frac{1}{\pi A_2} \int_{A_1} dA_1 \int dA_2 \left(\frac{\cos\theta_1 \cos\theta_2}{r_{12}^2}\right) . \tag{192}$$

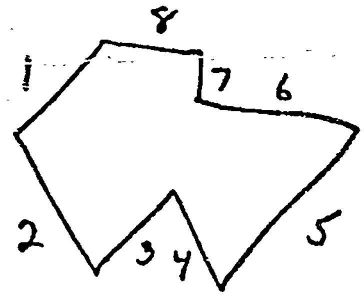
The F_{12} and F_{21} are dimensionless <u>view factors</u> - also called <u>shape</u> factors or <u>configuration</u> factors. The physical interpretation is that F_{12} gives the fraction of radiative energy emitted by body 1, which is intercepted by body 2. We note the symmetry relation

$$A_1 F_{12} = A_2 F_{21} , \tag{193}$$

and the inequality

$$0 \leq F_{12}, F_{21} \leq 1 . \tag{194}$$

Consider now n black bodies which form a complete enclosure:



bodies need not be "straight"

In the above picture, n = 8. Since they form a complete enclosure, we have

$$\sum_{i=1}^{n} F_{ji} = 1 \quad . \tag{195}$$

Let Q_i be the net radiative energy from body i per unit time. Then

$$Q_i = A_i \sigma T_i^4 - \sum_{j=1}^{n} A_j \sigma T_j^4 F_{ji} \quad . \tag{196}$$

Using

$$A_j F_{ji} = A_i F_{ij} \quad , \tag{197}$$

this becomes

$$Q_i = A_i \sigma [T_i^4 - \sum_{j=1}^{n} F_{ij} T_j^4] \quad . \tag{198}$$

Equation (198) represents n equations for 2n "unknowns", T_i and Q_i. Hence n of these unknowns must be specified, and the other n can be solved for. We note that any specification or solution, however, must satisfy

$$\sum_{i=1}^{n} Q_i = 0 \quad . \tag{199}$$

This follows from summing Eq. (196) over all i, making use of Eq. (195).

Three common uses of Eq. (198) are:

1. Compute Heat Fluxes

All the T_i are given, and the Q_i are evaluated from Eq. (198);

2. One Driving Temperature

One of the T_i, say T_1, is specified, and all walls except wall #1 are specified as insulated ($Q_i = 0$, $i \neq 1$). Equation (198) is used to solve for Q_1 and T_i, $i \neq 1$. We obviously find, in view of Eq. (199),

$$Q_1 = 0 \quad . \tag{200}$$

Since the equations are linear in T^4, we will also obtain

$$T_i^4 = K_i T_1^4 \tag{201}$$

where K_i, $i \neq 1$, depends upon all of the F_{ij} and the A_i.

3. Heat Transfer Between Two Surfaces

We specify T_1 and T_2, and all walls except 1 and 2 are specified as insulated. Equation (198) is used to solve for Q_1 and Q_2 and T_i, $i \neq 1,2$. In view of Eq. (199) we find

$$Q_2 = -Q_1 \tag{202}$$

and

$$Q_1 = \sigma A_1 F_{12}(T_1^4 - T_2^4) \tag{203}$$

where F_{12} depends upon all of the F_{ij} and the A_i.

Non-Black Bodies

This problem is more difficult than black bodies because of multiple reflection of radiation. To demonstrate this, consider two infinite slabs.

We have

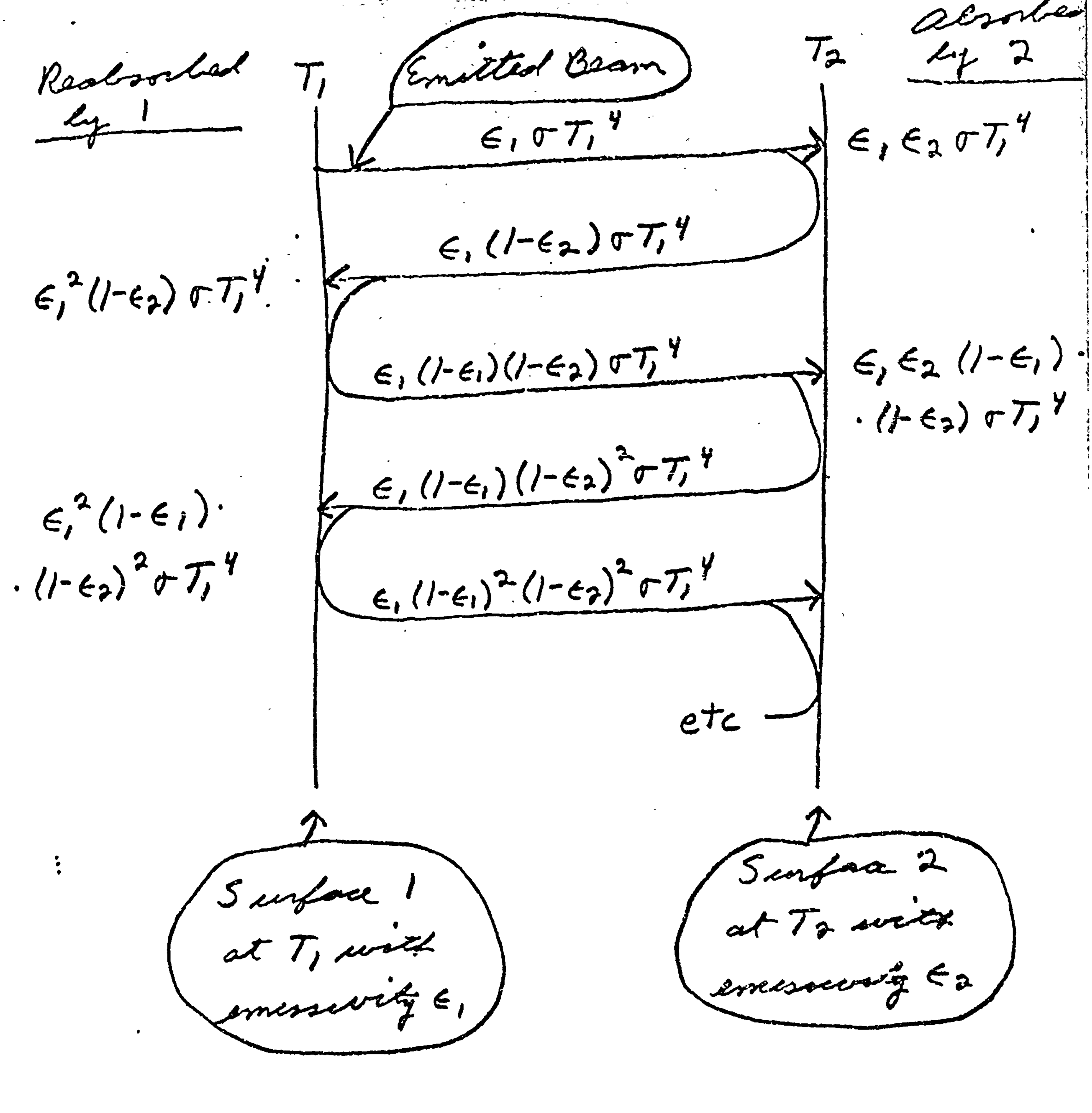
$$F_{12} = F_{21} = 1 \quad . \tag{204}$$

If the bodies were both black, then we would have

$$q_{12} = \text{net flux (per unit area) from body 1 to body 2}$$

$$= \sigma\left(T_1^4 - T_2^4\right) \quad . \tag{205}$$

Consider now grey bodies, with emissivities ε_1 and ε_2. We have the picture:


Reabsorbed by 1
T_1
Emitted Beam
T_2
absorbed by 2
$\epsilon_1 \sigma T_1^4$
$\epsilon_1 \epsilon_2 \sigma T_1^4$
$\epsilon_1 (1-\epsilon_2) \sigma T_1^4$
$\epsilon_1^2 (1-\epsilon_2) \sigma T_1^4$
$\epsilon_1 (1-\epsilon_1)(1-\epsilon_2) \sigma T_1^4$
$\epsilon_1 \epsilon_2 (1-\epsilon_1) \cdot (1-\epsilon_2) \sigma T_1^4$
$\epsilon_1 (1-\epsilon_1)(1-\epsilon_2)^2 \sigma T_1^4$
$\epsilon_1^2 (1-\epsilon_1) \cdot (1-\epsilon_2)^2 \sigma T_1^4$
$\epsilon_1 (1-\epsilon_1)^2 (1-\epsilon_2)^2 \sigma T_1^4$
etc
Surface 1 at T_1 with emissivity ϵ_1
Surface 2 at T_2 with emissivity ϵ_2

In drawing this picture, we have used Kirchoff's law, which says the probability of absorption is equal to the emissivity. Accounting for the absorption in wall 2 on each pass, we have:

$q_{\overrightarrow{12}}$ = flux (per unit area) absorbed by body 2 due to emission from body 1

$$= \sigma T_1^4[\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_2(1 - \varepsilon_1)(1 - \varepsilon_2)$$

$$+ \varepsilon_1\varepsilon_2(1 - \varepsilon_1)^2(1 - \varepsilon_2)^2$$

$$+ \varepsilon_1\varepsilon_2(1 - \varepsilon_1)^3(1 - \varepsilon_2)^3 + \ldots]$$

$$= \sigma T_1^4 \left[\frac{\varepsilon_1\varepsilon_2}{1-(1-\varepsilon_1)(1-\varepsilon_2)}\right] . \tag{206}$$

Similarly,

$$q_{\overrightarrow{21}} = \sigma T_2^4 \left[\frac{\varepsilon_1\varepsilon_2}{1-(1-\varepsilon_1)(1-\varepsilon_2)}\right] . \tag{207}$$

Hence the net flux (per unit area) passing from body 1 to body 2, q_{12}, is

$$q_{12} = q_{\overrightarrow{12}} - q_{\overrightarrow{21}} = \sigma(T_1^4 - T_2^4)\left[\frac{\varepsilon_1\varepsilon_2}{1-(1-\varepsilon_1)(1-\varepsilon_2)}\right] \tag{208}$$

Let us consider an enclosure of n bodies just as before, except now the bodies are grey rather than black. Define

q_i^+ = flux (per unit area) leaving the i^{th} surface.

q_i^- = flux (per unit area) impinging upon the i^{th} surface.

We have the statement that the energy impinging upon the ith surface is the sum of the contributions from all of the surfaces, i.e.,

$$A_i q_i^- = \sum_{j=1}^{n} A_j F_{ji} q_j^+ \quad . \tag{209}$$

Using

$$A_i F_{ij} = A_j F_{ji} \quad , \tag{210}$$

this becomes

$$q_i^- = \sum_{j=1}^{n} F_{ij} q_j^+ \quad . \tag{211}$$

Now, we have

$$q_i^+ = \underbrace{\varepsilon_i \sigma T_i^4}_{\text{emitted}} + \underbrace{(1 - \varepsilon_i) q_i^-}_{\text{reflected}} \quad , \tag{212}$$

and hence Eq. (211) becomes

$$q_i^- = \sum_{j=1}^{n} F_{ij} [\varepsilon_j \sigma T_j^4 + (1 - \varepsilon_j) q_j^-] \quad , \tag{213}$$

or, rewriting,

$$q_i^- - \sum_{j=1}^{n} F_{ij} (1 - \varepsilon_j) q_j^- = \sum_{j=1}^{n} F_{ij} \varepsilon_j \sigma T_j^4 \quad . \tag{214}$$

We can obtain an alternate form of this result in terms of the Q_i, defined as the net energy transfer from body i per unit time. We have

$$Q_i = A_i(q_i^+ - q_i^-) \quad , \tag{215}$$

and using Eq. (212) for q_i^+, this becomes

$$Q_i = A_i[\varepsilon_i \sigma T_i^4 - \varepsilon_i q_i^-] \quad , \tag{216}$$

or

$$q_i^- = \sigma T_i^4 - \frac{Q_i}{\varepsilon_i A_i} \quad . \tag{217}$$

Using this result in Eq. (214), we obtain as the generalization of the black body result:

$$Q_i - \sum_{j=1}^{n} F_{ij}(1 - \varepsilon_j) \frac{\varepsilon_i A_i}{\varepsilon_j A_j} Q_j$$

$$= \varepsilon_i A_i \sigma[T_i^4 - \sum_{j=1}^{n} F_{ij} T_j^4] \quad . \tag{218}$$

Just as in the black body case, one can easily show that Eq. (218) implies

$$\sum_{i=1}^{n} Q_i = 0 \quad . \tag{219}$$

We note that if all bodies are black ($\varepsilon_i = 1$), then Eq. (218) reduces to our previous result, Eq. (198).

If we consider two infinite planes, we have

$$F_{12} = F_{21} = 1 \; ; \quad F_{11} = F_{22} = 0 \quad . \tag{220}$$

If we set $A_1 = A_2 = 1$, then we have,

$$Q_1 = q_{12} \; ; \quad Q_2 = q_{21} \quad , \tag{221}$$

where q_{12} and q_{21} are the notation previously used in discussing the two parallel plane problem. Setting $i = 1$ in Eq. (218) then gives

$$q_{12} - (1 - \varepsilon_2) \frac{\varepsilon_1}{\varepsilon_2} q_{21} = \varepsilon_1 \sigma(T_1^4 - T_2^4) \quad . \tag{222}$$

But from Eqs. (219) and (221), we have

$$q_{21} = q_{12} \quad , \tag{223}$$

and Eq. (222) then yields

$$q_{12}[1 + (1 - \varepsilon_2) \frac{\varepsilon_1}{\varepsilon_2}] = \varepsilon_1 \sigma(T_1^4 - T_2^4) \quad , \tag{224}$$

which, when solved for q_{12}, gives

$$q_{12} = \sigma(T_1^4 - T_2^4)[\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 - \varepsilon_1 \varepsilon_2 + \varepsilon_2}] \quad . \tag{225}$$

This is the same result [see Eq. (208)], which we obtained by summing over an infinite number of reflections.

Just in the black body case, Eq. (218) represents n equations in 2n unknowns, T_i and Q_i. Thus n of these unknowns can be specified and the remaining n solved for, with Eq. (219) being one result of the solution (any specification must not violate this condition).

K. Relativistic Radiation Hydrodynamics

If the fluid speed u is an appreciable fraction of the speed of light c, it is necessary to formulate the equation of radiation hydrodynamics relativistically. These "u/c corrections" are important in many astrophysical applications and marginally important in nuclear weapons effects calculations. As we shall discuss later, there is one important case where, although u/c may formally be very small, it may be necessary to carry u/c terms to obtain the correct equations of radiation hydrodynamics.

We begin our discussion by considering the Lorentz transformation of the equation of transfer. We consider this equation as seen by an observer in an inertial frame of reference. We call this the zero frame and subscript all quantities with a zero. The equation of transfer is, rewriting Eq. (143) with zero subscripts,

$$\frac{1}{c}\frac{\partial I_o(\nu_o,\vec{\Omega}_o)}{\partial t_o} + \vec{\Omega}_o\cdot\vec{\nabla}_o I_o(\nu_o,\vec{\Omega}_o)$$

$$= \left[1 + \frac{c^2 I_o(\nu_o,\vec{\Omega}_o)}{2h\nu_o^3}\right]S_o(\nu_o,\vec{\Omega}_o) - \sigma_{ao}(\nu_o,\vec{\Omega}_o)I_o(\nu_o,\vec{\Omega}_o)$$

$$+ \int_0^\infty d\nu_o' \int_{4\pi} d\vec{\Omega}_o' \, \sigma_{so}(\nu_o'\to\nu_o,\vec{\Omega}_o'\to\vec{\Omega}_o)\frac{\nu_o}{\nu_o'} I_o(\nu_o',\vec{\Omega}_o')\left[1 + \frac{c^2 I_o(\nu_o,\vec{\Omega}_o)}{2h\nu_o^3}\right]$$

$$- \int_0^\infty d\nu_o' \int_{4\pi} d\vec{\Omega}_o' \, \sigma_{so}(\nu_o\to\nu_o',\vec{\Omega}_o\to\vec{\Omega}_o')I_o(\nu_o,\vec{\Omega}_o)\left[1 + \frac{c^2 I_o(\nu_o',\vec{\Omega}_o')}{2h\nu_o'^3}\right] \quad (226)$$

Equation (226) is a slight generalization of Eq. (143) in that the source S and absorption coefficient are allowed an $\vec{\Omega}$ dependence and the scattering kernel σ_s can depend upon $\vec{\Omega}$ and $\vec{\Omega}'$ separately. The same generalization of Eq. (147) is

$$\frac{1}{c}\frac{\partial I_o(\nu_o,\vec{\Omega}_o)}{\partial t_o} + \vec{\Omega}_o\cdot\vec{\nabla} I_o(\nu_o,\vec{\Omega}_o)$$

$$= \sigma'_{ao}(\nu_o,\vec{\Omega}_o)[B_o(\nu_o,\vec{\Omega}_o) - I_o(\nu_o,\vec{\Omega}_o)]$$

$$+ \int_0^\infty d\nu'_o \int_{4\pi} d\vec{\Omega}'_o\ \sigma_{so}(\nu'_o\to\nu_o,\vec{\Omega}'_o\to\vec{\Omega}_o)\,\frac{\nu_o}{\nu'_o}\, I_o(\nu'_o,\vec{\Omega}'_o)\left[1 + \frac{c^2 I_o(\nu_o,\vec{\Omega}_o)}{2h\nu_o^3}\right]$$

$$- \int_0^\infty d\nu'_o \int_{4\pi} d\vec{\Omega}'_o\ \sigma_{so}(\nu_o\to\nu'_o,\vec{\Omega}_o\to\vec{\Omega}'_o) I_o(\nu_o,\vec{\Omega}_o)\left[1 + \frac{c^2 I_o(\nu'_o,\vec{\Omega}'_o)}{2h\nu'^3_o}\right] . \quad (227)$$

In both Eqs. (226) and (227) we have not explicitly written the $\vec{r}$ and t arguments of all quantities, but these dependences are understood.

We now consider a second inertial frame of reference moving with velocity $\vec{v}$ with respect to the zero frame. In this second frame we leave all quantities unadorned. Hence to an observer in this frame, Eq. (226) or Eq. (227) is the equation of transfer, with all zero subscripts dropped. By demanding that the equation of transfer be invariant under a Lorentz transformation, we can relate all of the components in the equations in the two frames. Omitting the details, the results are:

If we define

$$\lambda = (1 - v^2/c^2)^{-1/2} \quad , \qquad (228)$$

$$D = 1 + \vec{\Omega}\cdot\vec{v}/c \quad , \qquad (229)$$

$$D' = 1 + \vec{\Omega}'\cdot\vec{v}/c \quad , \qquad (230)$$

then we have the transformations:

$$\nu_o = \lambda D \nu \quad , \tag{231}$$

$$\vec{\Omega}_o = \frac{1}{\lambda D} \left[\vec{\Omega} + \frac{\lambda}{\lambda + 1} (\lambda D + 1) \frac{\vec{v}}{c}\right] \quad , \tag{232}$$

$$\frac{1}{c} \frac{\partial}{\partial t_o} + \vec{\Omega}_o \cdot \vec{\nabla}_o = \frac{1}{\lambda D} \left[\frac{1}{c} \frac{\partial}{\partial t} + \vec{\Omega} \cdot \vec{\nabla}\right] \quad , \tag{233}$$

$$1 - \vec{\Omega}_o \cdot \vec{\Omega}'_o = \frac{1}{\lambda^2 DD'} (1 - \vec{\Omega} \cdot \vec{\Omega}') \quad , \tag{234}$$

$$I_o(\nu_o, \vec{\Omega}_o) = (\lambda D)^3 I(\nu, \vec{\Omega}) \quad , \tag{235}$$

$$S_o(\nu_o, \vec{\Omega}_o) = (\lambda D)^2 S(\nu, \vec{\Omega}) \quad , \tag{236}$$

$$\sigma_{ao}(\nu_o, \vec{\Omega}_o) = \frac{1}{\lambda D} \sigma a(\nu, \vec{\Omega}) \quad , \tag{237}$$

$$B_o(\nu_o, \vec{\Omega}_o) = (\lambda D)^3 B(\nu, \vec{\Omega}) \quad , \tag{238}$$

$$\sigma'_{ao}(\nu_o, \vec{\Omega}_o) = \frac{1}{\lambda D} \sigma'_a(\nu, \vec{\Omega}) \quad , \tag{239}$$

$$\sigma_{so}(\nu_o \rightarrow \nu'_o, \vec{\Omega}_o \rightarrow \vec{\Omega}'_o) = \frac{D'}{D} \sigma_s(\nu \rightarrow \nu', \vec{\Omega} \rightarrow \vec{\Omega}') \quad , \tag{240}$$

$$d\nu_o d\vec{\Omega}_o = \frac{1}{\lambda D} d\nu d\vec{\Omega} \quad . \tag{241}$$

We now use these results to account for relativistic effects in the equation of transfer. It was pointed out earlier that there is no preferred direction in the fluid for radiation hydrodynamic problems. Hence σ_a, σ'_a, S, and B, which describe the absorption and source of photons should be independent of $\vec{\Omega}$, the flight direction of a photon. Further, σ_s the scattering kernel, should only depend upon the scattering angle, whose cosine is $\vec{\Omega}\cdot\vec{\Omega}'$, rather than upon the directions $\vec{\Omega}$ and $\vec{\Omega}'$ separately. However, the fact that in radiation hydrodynamic problems the fluid is in general in motion changes the situation. The fluid is still isotropic, but, as seen by an inertial frame observer, this motion does introduce a preferred direction in the matter, namely the direction of motion of the fluid. This in turn, in the relativistic limit, introduces an $\vec{\Omega}$ dependence in σ_a, σ_a', S, and B, and separate $\vec{\Omega}$ and $\vec{\Omega}'$ dependences in σ_s.

Taking this into account, the equation of transfer, Eq. (143), should be written

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = S(\nu,\vec{\Omega})\left[1 + \frac{c^2 I(\nu,\vec{\Omega})}{2h\nu^3}\right] - \sigma_a(\nu,\vec{\Omega})I(\nu,\vec{\Omega})$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\, \frac{\nu}{\nu'}\, \sigma_s(\nu'\rightarrow\nu,\vec{\Omega}'\rightarrow\vec{\Omega})I(\nu',\vec{\Omega}')\left[1 + \frac{c^2 I(\nu,\vec{\Omega})}{2h\nu^3}\right]$$

$$- \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\, \sigma_s(\nu\rightarrow\nu',\vec{\Omega}\rightarrow\vec{\Omega}')I(\nu,\vec{\Omega})\left[1 + \frac{c^2 I(\nu',\vec{\Omega}')}{2h\nu'^3}\right] \quad , (242)$$

and Eq. (147) becomes

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \sigma_a'(\nu,\vec{\Omega})[B(\nu,\vec{\Omega}) - I(\nu,\vec{\Omega})]$$

$$+ \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\, \frac{\nu}{\nu'}\, \sigma_s(\nu'\rightarrow\nu,\vec{\Omega}'\rightarrow\vec{\Omega})I(\nu',\vec{\Omega}')\left[1 + \frac{c^2 I(\nu,\vec{\Omega})}{2h\nu^3}\right]$$

$$- \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}'\, \sigma_s(\nu\rightarrow\nu',\vec{\Omega}\rightarrow\vec{\Omega}')I(\nu,\vec{\Omega})\left[1 + \frac{c^2 I(\nu',\vec{\Omega}')}{2h\nu'^3}\right] \quad . (243)$$

It should be emphasized that these additional angular dependences are not inherent properties of the fluid, but arise only from the relative motion between the fluid and the observer. Hence we can use the Lorentz transformation results just presented to compute these dependences. This is argued as follows:

We consider an observer in an inertial frame of reference observing radiative transfer in a moving fluid. At a particular space point $\vec{r}$ and time t the fluid has a macroscopic velocity $\vec{u}(\vec{r},t)$ as seen by this observer. We call the frame of reference in which the fluid is at rest the zero frame, and the frame of the observer the unadorned frame. The transformation velocity $\vec{v}$ of the Lorentz transformation of Eqs. (228) through (241) is then

$$\vec{v} = -u(\vec{r},t) \quad . \tag{244}$$

Since the observer is in an inertial frame of reference, Eq. (242) or Eq. (243) is the appropriate transport equation. We assume the source functions S and B, the absorption coefficients σ_a and σ_a', and the scattering kernel σ_s are known in the zero frame (the fluid rest frame), and use the Lorentz transformation results to obtain these functions in the unadorned frame.

It must be noted that the zero frame, defined as the frame for which the fluid is at rest, is not in general an inertial frame since the fluid velocity is a function of both space and time. Hence Eq. (226) or (227), valid only in an inertial frame, is not a proper description of radiative transfer in the fluid rest frame. More to the point, the fact that the fluid rest frame is not an inertial frame implies that the Lorentz transformation between frames cannot be used. However, certain of these transformations can be used in the present context. That is, even though the fluid rest frame is not an inertial frame, one can envision an inertial frame that instantaneously, at time t and space point $\vec{r}$, coincides with the fluid rest frame. Since the source terms (by source terms here we mean all terms except the streaming terms, i.e., the emission source; absorption; and scattering terms) in the equation of transfer are well defined at a single time t and space point $\vec{r}$ (i.e., t and $\vec{r}$ are only parameters in the source terms; no operators involving time or space appear in these terms), the Lorentz transformation can indeed by used to relate the source terms in the fluid rest frame to those in the unadorned (observer) frame. The streaming terms, on the other hand, involve derivatives with respect to space and time. To define these derivatives via a Lorentz transformation the fluid rest frame must be an inertial frame for an arbitrarily small, but nonzero, interval of space and time. Since, in general, such an interval does not exist for a fluid in non-uniform motion, the Lorentz transformation is not valid for the streaming terms. Fortunately, we need concern ourselves here only with the source terms since we wish to write the equation of transfer in the unadorned frame and in this frame, since it is inertial, the streaming terms are already known.

We first consider the terms S, B, σ_a, and σ_a'. Since the fluid is isotropic, all of these functions are independent of $\vec{\Omega}$ and hence depend only upon frequency in the fluid rest (zero) frame. From Eqs. (228), (229), (231), (236), and (244) we obtain

$$S(\nu,\vec{\Omega}) = \frac{1}{(\Lambda E)^2} S_o(\nu_o) \quad , \tag{245}$$

where

$$\nu_o = \Lambda E \nu \quad , \tag{246}$$

$$\Lambda = (1 - u^2/c^2)^{-1/2} \quad , \tag{247}$$

$$E = 1 - \vec{\Omega}\cdot\vec{u}/c \quad . \tag{248}$$

For u/c << 1, it is sensible to expand $S(\nu,\vec{\Omega})$ in powers of u/c. Correct to first order, we find

$$S(\nu,\vec{\Omega}) = S_o(\nu) + (\vec{\Omega}\cdot\frac{\vec{u}}{c})[2S_o(\nu) - \nu\frac{dS_o(\nu)}{d\nu}] \quad . \tag{249}$$

Similarly, from Eqs. (237), (238), and (239) we obtain

$$B(\nu,\vec{\Omega}) = \frac{1}{(\Lambda E)^3} B_o(\nu_o) \quad , \tag{250}$$

$$\sigma_a(\nu,\vec{\Omega}) = (\Lambda E)\sigma_{ao}(\nu_o) \quad , \tag{251}$$

$$\sigma'_a(\nu,\vec{\Omega}) = (\Lambda E)\sigma'_{ao}(\nu_o) \quad , \tag{252}$$

which, correct to the first order, gives

$$B(\nu,\vec{\Omega}) = B_o(\nu) + (\vec{\Omega}\cdot\frac{\vec{u}}{c})[3B_o(\nu) - \nu\frac{dB_o(\nu)}{d\nu}] \quad , \tag{253}$$

$$\sigma_a(\nu,\vec{\Omega}) = \sigma_{ao}(\nu) - (\vec{\Omega}\cdot\frac{\vec{u}}{c})[\sigma_{ao}(\nu) + \nu\frac{d\sigma_{ao}(\nu)}{d\nu}] \quad , \tag{254}$$

$$\sigma'_a(\nu,\vec{\Omega}) = \sigma'_{ao}(\nu) - (\vec{\Omega}\cdot\frac{\vec{u}}{c})[\sigma'_{ao}(\nu) + \nu\frac{d\sigma'_{ao}(\nu)}{d\nu}] \quad , \tag{255}$$

In addition, from Eq. (240) we can deduce the scattering kernal in the unadorned frame. We have

$$\sigma_s(\nu\to\nu',\vec{\Omega}\to\vec{\Omega}') = \frac{E}{E'}\,\sigma_{so}(\nu_o\to\nu'_o,\vec{\Omega}_o\cdot\vec{\Omega}'_o) \quad , \tag{256}$$

where E is given by Eq. (248) and

$$E' = 1 - \vec{\Omega}'\cdot\vec{u}/c \quad . \tag{257}$$

In writing Eq. (256) we have explicitly shown that in the zero frame the scattering kernel depends only upon the scattering angle. For small $\vec{u}/c$, Eq. (256) yields

$$\sigma_s(\nu\rightarrow\nu',\vec{\Omega}\rightarrow\vec{\Omega}') = \sigma_{so}(\nu\rightarrow\nu',\vec{\Omega}\cdot\vec{\Omega}')$$

$$-\left(\vec{\Omega}\cdot\frac{\vec{u}}{c}\right)\left[\sigma_{so} + \nu\frac{d\sigma_{so}}{d\nu} + (1-\vec{\Omega}\cdot\vec{\Omega}')\frac{d\sigma_{so}}{d(\vec{\Omega}\cdot\vec{\Omega}')}\right]$$

$$-\left(\vec{\Omega}'\cdot\frac{\vec{u}}{c}\right)\left[-\sigma_{so} + \nu'\frac{d\sigma_{so}}{d\nu'} + (1-\vec{\Omega}\cdot\vec{\Omega}')\frac{d\sigma_{so}}{d(\Omega\cdot\Omega')}\right] \quad . \tag{258}$$

As a concrete example of putting these considerations together, we consider the simple case of an equation of transfer, neglecting induced effects, in which the scattering is grey (the scattering cross section is independent of frequency), isotropic and coherent, i.e.,

$$\sigma_{so}(\vec{\nu}_o'\rightarrow\vec{\nu}_o,\vec{\Omega}\cdot\vec{\Omega}') = \frac{\sigma_{so}}{4\pi}\,\delta(\nu_o' - \nu_o) \quad . \tag{259}$$

In the absence of $\vec{u}/c$ terms, the equation of transfer in the inertial (unadorned) frame would be written

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + [\sigma_{ao}(\nu) + \sigma_{so}]I(\nu,\vec{\Omega})$$

$$= S_o(\nu) + \frac{\sigma_{so}}{4\pi}\int_{4\pi} d\vec{\Omega}' I(\nu,\vec{\Omega}') \quad , \tag{260}$$

or, introducing,

$$E(\nu) \equiv \frac{1}{c}\int_{4\pi} d\vec{\Omega}' I(\nu,\vec{\Omega}') \quad , \tag{261}$$

Eq. (260) can be written

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + [\sigma_{ao}(\nu) + \sigma_{so}]I(\nu,\vec{\Omega})$$

$$= S_o(\nu) + \frac{\sigma_{so}}{4\pi}\, cE(\nu) \quad . \tag{262}$$

With the inclusion of relativistic terms, correct to order $\vec{u}/c$, Eq. (260) becomes

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + [\sigma_{ao}(\nu) + \sigma_{so}]I(\nu,\vec{\Omega})$$

$$= S_o(\nu) + \frac{\sigma_{so}}{4\pi}\int_{4\pi} d\vec{\Omega}'I(\nu,\vec{\Omega}')$$

$$+ (\vec{\Omega}\cdot\frac{\vec{u}}{c})[\sigma_{ao}(\nu) + \sigma_{so} + \nu\,\frac{d\sigma_{ao}(\nu)}{d\nu}]I(\nu,\vec{\Omega})$$

$$+ (\vec{\Omega}\cdot\frac{\vec{u}}{c})[2S_o(\nu) - \nu\,\frac{dS_o(\nu)}{d\nu}]$$

$$- \frac{\sigma_{so}}{4\pi}\int_{4\pi} d\vec{\Omega}'(\vec{\Omega}'\cdot\frac{\vec{u}}{c})[I(\nu,\vec{\Omega}') - \nu\,\frac{\partial I(\nu,\vec{\Omega}')}{\partial \nu}]$$

$$+ \frac{\sigma_{so}}{4\pi}(\vec{\Omega}\cdot\frac{\vec{u}}{c})\int_{4\pi} d\vec{\Omega}'[2I(\nu,\vec{\Omega}') - \nu\,\frac{\partial I(\nu,\vec{\Omega}')}{\partial \nu}] \quad , \tag{263}$$

or, using Eq. (261) and introducing

$$\vec{F}(\nu) = \int_{4\pi} d\vec{\Omega}'\vec{\Omega}'\; I(\nu,\vec{\Omega}') \quad , \tag{264}$$

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) + [\sigma_{ao}(\nu) + \sigma_{so}]I(\nu,\vec{\Omega})$$

$$= S_o(\nu) + \frac{\sigma_{so}}{4\pi}cE(\nu) + (\vec{\Omega}\cdot\frac{\vec{u}}{c})[\sigma_{ao}(\nu) + \sigma_{so} + \nu\frac{d\sigma_{ao}(\nu)}{d\nu}]I(\nu,\vec{\Omega})$$

$$+ (\vec{u}\cdot\frac{\vec{u}}{c})[2S_o(\nu) - \nu\frac{dS_o(\nu)}{d\nu}] - \frac{\sigma_{so}}{4\pi}[\frac{\vec{u}}{c}\cdot\vec{F}(\nu) - \nu\frac{\vec{u}}{c}\cdot\frac{\partial\vec{F}}{\partial\nu}]$$

$$+ \frac{\sigma_{so}}{4\pi}(\vec{\Omega}\cdot\frac{\vec{u}}{c})[2cE(\nu) - c\nu\frac{\partial E(\nu)}{\partial\nu}] \quad . \tag{265}$$

[The $\partial E/\partial\nu$ and $\partial\vec{F}/\partial\nu$ terms arise from integrating by parts to eliminate derivatives of delta functions that arise from using Eq. (259) in Eq. (258)]. We note the complexity that has been introduced by retaining terms of order $\vec{u}/c$.

We now turn to the question of relative hydrodynamics. The relativistic hydrodynamic equations in the absence of a radiation field are given in many texts on relativity and fluid mechanics. The approach universably used to derive these equations is to employ the energy-momentum tensor. This tensor is obtained by arguing that it must have a certain form to undergo the proper Lorentz transformation and to reduce to the correct diagonal tensor for a fluid at rest. In our discussion of the relativistic hydrodynamic equations including radiative contributions we use kinetic theory arguments rather than an energy-momentum tensor containing radiation terms. This avoids the use of the transformation properties of tensors and seems to be a more basic starting point. In particular, our discussion emphasizes the assumption needed to obtain a hydrodynamic description of the motion of an ideal fluid. Further, the concept of fluid pressure enters naturally.

We consider a fluid composed of particles of rest mass m_o having various momenta $\vec{p}$ and described microscopically at time t by a distribution function per unit volume and per unit momentum

$\psi(\vec{r},t,\vec{p})$. For simplicity, we drop all arguments $\vec{r}$ and t and simply denote this distribution function by $\psi(\vec{p})$. Thus the number of particles at time t in a differential volume element $d\vec{r}$ centered at $\vec{r}$ and in a differential momentum element $d\vec{p}$ centered at $\vec{p}$ is given by $\psi(\vec{p})d\vec{r}d\vec{p}$. The macroscopic velocity of the fluid is denoted by $\vec{u}$; i.e., if $\vec{v}$ denotes the velocity of a particle with momentum $\vec{p}$, then

$$\vec{u} = \frac{\int d\vec{p}\ \vec{v}\ \psi(\vec{p})}{\int d\vec{p}\ \psi(\vec{p})} \quad . \tag{266}$$

That is, the fluid velocity $\vec{u}$ is just the velocity of the individual particles that make up the fluid averaged over the distribution function.

We introduce a second frame of reference, namely the frame moving with the fluid. We refer to this as the fluid rest frame and subscript all quantities in this frame of reference with a zero. In particular, we denote by $\psi_o(\vec{p}_o)$ the particle distribution function in the rest frame, with $\vec{p}_o$ denoting the momentum in this frame. It is important to note that the fluid rest frame is not in general an inertial frame of reference since the fluid can undergo accelerations at any point in space and time.

The basic assumption that leads to ideal fluid hydrodynamics is that the momentum dependence of the distribution function is isotropic in the fluid rest frame. That is, $\psi_o(\vec{p}_o)$ depends upon only the magnitude of the momentum, and not its direction.

In the fluid rest frame, we define three macroscopic quantities N_o, E_{to}, and P_{mo} by the equations

$$N_o = \int d\vec{p}_o \psi_o(\vec{p}_o) \quad , \tag{267}$$

$$E_{to} = \int d\vec{p}_o E_o \psi_o(\vec{p}_o) \quad , \tag{268}$$

$$P_{mo} = \int d\vec{p}_o (\vec{v}_o \cdot \vec{n})(\vec{p}_o \cdot \vec{n}) \psi_o(\vec{p}_o) \quad . \tag{269}$$

Here E_o is energy, including the rest energy, asociated with a particle of momentum $\vec{p}_o$; and $\vec{v}_o$ is the velocity associated with a particle of momentum $\vec{p}_o$. The vector $\vec{n}$ is an arbitrary unit vector - since $\psi_o(\vec{p}_o)$ is isotropic by assumption, P_{mo} does not depend upon the choice of $\vec{n}$. The physical interpretation of N_o and E_{to} is immediate. N_o is just the particle density and E_{to} is the total energy density, both in the fluid rest frame. From its definition, P_{mo} is just the rate of transfer of the momentum component parallel to $\vec{n}$ across a surface of unit area whose normal direction is $\vec{n}$. This quantity is conventionally called the material pressure, again defined in the fluid rest frame.

We now define the six quantities needed to derive the hydrodynamic equations, namely the number, momentum, and energy densities and fluxes, all in the unadorned, or observer frame (which is an inertial frame). We have

$$\text{Number Density} \equiv \int d\vec{p} \; \psi(\vec{p}) \quad , \tag{270}$$

$$\text{Momentum Density} \equiv \int d\vec{p} \; \vec{p} \; \psi(\vec{p}) \quad , \tag{271}$$

$$\text{Energy Density} \equiv \int d\vec{p} \; E \; \psi(\vec{p}) \quad , \tag{272}$$

$$\text{Number Flux} \equiv \int d\vec{p} \; \vec{v} \; \psi(\vec{p}) \quad , \tag{273}$$

$$\text{Momentum Flux} \equiv \int d\vec{p} \; \vec{v} \; \vec{p} \; \psi(\vec{p}) \quad , \tag{274}$$

$$\begin{matrix}\text{Energy} \\ \text{Flux}\end{matrix} \equiv \int d\vec{p}\ \vec{v}\ E\ \psi(\vec{p}) \quad . \tag{275}$$

We evaluate these six integrals by changing variables from $\vec{p}$ to $\vec{p}_o$, i.e., we perform the integrations in the fluid rest frame. The results are that these six quantities in the observer frame can be expressed in terms of the three quantities in the fluid rest frame given by Eqs. (267) through (269), in addition to the fluid velocity $\vec{u}$ and the relativistic factor

$$\Lambda \equiv (1 - u^2/c^2)^{-1/2} \quad . \tag{276}$$

We sketch the details for Eq. (270), and merely quote the other five results. We have

$$N \equiv \int d\vec{p}\ \psi(\vec{p}) \quad . \tag{277}$$

To evaluate this integral, we change variables of integration from $\vec{p}$ to $\vec{p}_o$, the rest frame momentum. Since $\vec{p}$ and iE/c form a four vector, they transform according to the usual Lorentz transformation. The result is

$$\vec{p} = \vec{p}_o + [\frac{(\vec{u}\cdot\vec{p}_o)(\Lambda-1)}{u^2} + \frac{\Lambda E_o}{c^2}]\vec{u} \quad , \tag{278}$$

$$E = \Lambda(E_o + \vec{u}\cdot\vec{p}_o) \quad . \tag{279}$$

The variable $\vec{p}$ and E are not independent, but are related by

$$E^2 = p^2c^2 + m_o^2c^4 \quad , \tag{280}$$

with a similar relationship valid in the zero frame. From Eqs. (278) through (280) one can compute the Jacobian between $\vec{p}$ and $\vec{p}_o$ that relates the differentials $d\vec{p}$ and $d\vec{p}_o$. The result is

$$d\vec{p} = \frac{E}{E_o}\, d\vec{p}_o \quad . \tag{281}$$

Futher, it is well known that the distribution function is a Lorentz invariant, i.e.,

$$\psi(\vec{p}) = \psi_o(\vec{p}_o) \quad . \tag{282}$$

Using Eqs. (281) and (282) in Eq. (277) gives

$$N = \int d\vec{p}_o\, \Lambda[1 + (\vec{u}\cdot\vec{p}_o)/E_o]\psi_o(\vec{p}_o) \quad , \tag{283}$$

where we have used Eq. (279) for the ratio E/E_o. Since $\psi_o(\vec{p}_o)$ is isotropic (or, more generally, by definition of the fluid rest frame), the term involving $\vec{p}_o$ in Eq. (283) has a zero integral and hence, recalling Eq. (267),

$$N = \Lambda N_o \quad . \tag{284}$$

In deriving this result, we have used the Lorentz transformation to transform to the fluid rest frame. As we noted earlier, however, the fluid rest frame is not an inertial frame. Nevertheless, the Lorentz transformation can properly be used, as we argued in connection with our discussion of the equation of transfer.

Similar manipulations allow us to evaluate the other five integrals defined by Eqs. (271) through (275). We find, including Eq. (284), the six results in the unadorned (observer) frame:

$$\text{Number density} = \Lambda N_o \quad , \tag{285}$$

$$\text{Momentum density} = \frac{\Lambda^2}{c^2}\,(E_{to} + P_{mo})\vec{u} + \frac{\vec{F}}{c^2} \quad , \tag{286}$$

$$\text{Energy density} = \Lambda^2(E_{to} + P_{mo}) - P_{mo} + E \quad , \tag{287}$$

$$\text{Number flux} = \Lambda N_o\vec{u} \quad , \tag{288}$$

$$\text{Momentum flux} = P_{mo}\overleftrightarrow{I} + \frac{\Lambda^2}{c^2}\,(E_{to} + P_{mo})\vec{u}\vec{u} + \overleftrightarrow{P} \quad , \tag{289}$$

$$\text{Energy flux} = \Lambda^2(E_{to} + P_{mo})\vec{u} + \vec{F} \quad . \tag{290}$$

In writing Eqs. (285) through (290), we have included the radiative contributions where E (the radiation energy density) $\vec{F}$ (the radiative flux) and $\overleftrightarrow{P}$ (the radiation pressure tensor) are defined by Eqs. (21) through (23).

With these results, it is straightforward to derive the Eulerian equations of fluid dynamics, including effects of a radiation field. We let $D(\vec{r},t)$ represent the density of the quantity under consideration (particle number, momentum, or energy) and let $\vec{F}(\vec{r},t)$ denote the corresponding flux. The conservation equation is simply

$$\frac{\partial D}{\partial t} + \vec{\nabla}\cdot\vec{F} = 0 \quad . \tag{291}$$

(If an external source is present, the rhs of this equation would not be zero.) Applying Eq. (291) to the three conserved quantities, namely particle number, momentum, and energy, we obtain, using Eqs. (285) through (290) for the relevant densities and fluxes,

$$\frac{\partial(\Lambda N_o)}{\partial t} + \vec{\nabla}\cdot(\Lambda N_o\vec{u}) = 0 \quad , \tag{292}$$

$$\frac{\partial}{\partial t}\left[\frac{\Lambda^2}{c^2}(E_{to}+P_{mo})\vec{u}+\frac{\vec{F}}{c^2}\right]+\vec{\nabla}P_{mo}$$

$$+\vec{\nabla}\cdot\left[\frac{\Lambda^2}{c^2}(E_{to}+P_{mo})\vec{u}\vec{u}+\vec{P}\right]=0 \quad , \tag{293}$$

$$\frac{\partial}{\partial t}\left[\Lambda^2(E_{to}+P_{mo})-P_{mo}+E\right]+\vec{\nabla}\cdot\left[\Lambda^2(E_{to}+P_{mo})\vec{u}+\vec{F}\right]=0 \; . \tag{294}$$

Equations (292) through (294) are the Eulerian form of the relativistic ideal fluid equations.

We can put these equations in a somewhat more useful form by redefining some of the variables. In particular, we eliminate N_o in these equations in favor of ρ_o, defined as

$$\rho_o = m_o N_o \quad . \tag{295}$$

The quantity ρ_o is just the rest frame density. We also define E_{mo} as

$$E_{mo} = E_{to} - \rho_o c^2 \quad , \tag{296}$$

so that E_{mo} is the fluid energy density (in the fluid rest frame) in excess of the rest energy. Then Eqs. (292) through (294) become:

$$\frac{\partial(\Lambda\rho_o)}{\partial t}+\vec{\nabla}\cdot(\Lambda\rho_o\vec{u})=0 \quad , \tag{297}$$

$$\frac{\partial}{\partial t}\left[\frac{\Lambda^2}{c^2}(\rho_o c^2+E_{mo}+P_{mo})\vec{u}+\frac{\vec{F}}{c^2}\right]+\vec{\nabla}P_{mo}$$

$$+\vec{\nabla}\cdot\left[\frac{\Lambda^2}{c^2}(\rho_o c^2+E_{mo}+P_{mo})\vec{u}\vec{u}+\vec{P}\right]=0 \quad , \tag{298}$$

$$\frac{\partial}{\partial t}\left[\Lambda^2(\rho_o c^2 + E_{mo} + P_{mo}) - P_{mo} + E\right]$$

$$+ \vec{\nabla}\cdot\left[\Lambda^2(\rho_o c^2 + E_{mo} + P_{mo})\vec{u} + \vec{F}\right] = 0 \quad . \tag{299}$$

An alternate form of the energy equation follows by multiplying Eq. (297) by c^2 and subtracting the result from Eq. (299). This has the effect of deleting the particle rest energy contribution from the energy equation and makes the passage to the nonrelativistic limit easier. Hence an equivalent set of relativistic hydrodynamic equations, our final form for the Eulerian equation, is:

$$\frac{\partial(\Lambda\rho_o)}{\partial t} + \vec{\nabla}\cdot(\Lambda\rho_o\vec{u}) = 0 \quad , \tag{300}$$

$$\frac{\partial}{\partial t}\left[\frac{\Lambda^2}{c^2}(\rho_o c^2 + E_{mo} + P_{mo})\vec{u} + \frac{\vec{F}}{c^2}\right] + \vec{\nabla}P_{mo}$$

$$+ \vec{\nabla}\cdot\left[\frac{\Lambda^2}{c^2}(\rho_o c^2 + E_{mo} + P_{mo})\vec{u}\vec{u} + \vec{P}\right] = 0 \quad , \tag{301}$$

$$\frac{\partial}{\partial t}\left[\Lambda(\Lambda - 1)\rho_o c^2 + \Lambda^2(E_{mo} + P_{mo}) - P_{mo} + E\right]$$

$$+ \vec{\nabla}\cdot\left[\Lambda(\Lambda - 1)\rho_o c^2\vec{u} + \Lambda^2(E_{mo} + P_{mo})\vec{u} + \vec{F}\right] = 0 \quad . \tag{302}$$

In the limit $\vec{u}/c \rightarrow 0$, Eqs. (300) through (302) reduce to the nonrelativistic equations, Eqs. (16) through (18).

III. APPROXIMATE MODELS OF RADIATIVE TRANSFER

The equation of radiative transfer is obviously quite complex; the specific intensity, which is the dependent variable in this equation, depends in general upon seven independent variables $(\vec{r}, \nu, \vec{\Omega}, t)$. Even in the simplest physically interesting situation of time and frequency independent transport in plane geometry (then only two independent variables z and μ are involved), one can obtain analytic solutions in only a very small number of limiting cases. Hence in general one must approximate the equation of transfer, either analytically or numerically, in order to obtain a solution.

Most approximate descriptions of radiative transfer are based upon the integro-differential equation rather than the integral equation. The frequency and angle dependences of the specific intensity, which give rise to the integral terms in this equation, are generally approximated analytically. This leads to a finite (ane hopefully small) number of coupled differential equations in the space and time variables. These equations are then conventionally solved numerically via more or less standard finite difference techniques.

We discuss here a limited number of analytic approximations employed in frequency and angle. The methods we shall consider certainly do not represent the totality of all methods that can, and have, been used in radiation hydrodynamics calculations. They are, however, the techniques most commonly used in practice. The finite difference methods used in space and time will not be considered. Such techniques, especially with the advent of high speed computers, can be very sophisticated and represent a discipline within themselves. For simplicity of exposition, we consider the simple case of an equation of transfer which neglects induced effects, and involving scattering which is both isotropic and coherent. The equation of transfer is then

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I + \sigma I = \frac{c}{4\pi}\left(\sigma_a b + \sigma_s E\right) \quad , \tag{303}$$

where

$$b = \frac{4\pi B}{c} \tag{304}$$

is the <u>energy density</u> associated with the Planck distribution, and E is the radiation field energy density

$$E = \frac{1}{c} \int_{4\pi} d\vec{\Omega} I(\vec{\Omega}) \quad . \tag{305}$$

Integration of Eq. (303) over all solid angle yields the conservation equation

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{F} = c\sigma_a (b - E) \quad , \tag{306}$$

where $\vec{F}$ is the radiation flux

$$\vec{F} = \int_{4\pi} d\vec{\Omega} \vec{\Omega} I(\vec{\Omega}) \quad . \tag{307}$$

A. The Eddington or Diffusion Approximation

For Eq. (306) to be useful, we need a second relationship that gives $\vec{F}$ as a functional of E. The basic assumption underlying the classical diffusion, or Eddington, description of radiative transfer is that the angular dependence of the specific intensity can be represented by the first two terms in a spherical harmonic expansion. That is, it is assumed that

$$I(\vec{\Omega}) = \frac{1}{4\pi} \left[cE + 3\vec{\Omega} \cdot \vec{F} \right] \quad . \tag{308}$$

Use of Eq. (308) in Eq. (303), multiplication of the result by $\vec{\Omega}$, and the subsequent integration over $\vec{\Omega}$ yields

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + \frac{1}{3}\vec{\nabla}(cE) + \sigma\vec{F} = 0 \quad . \tag{309}$$

Equations (306) and (309) form a closed set of equations for E and $\vec{F}$, of the telegraphers form. They yield a finite speed of propagation, but of value $c/\sqrt{3}$, rather than the correct value c. This approximation is often referred to as the P-1 approximation.

For these equations to reduce to a diffusion-like description of radiative transfer, we must demand, to be consistent with normal usage of the term diffusion, that Eq (309) reduce to a Fick's law of diffusion, i.e.,

$$\vec{F} = -D\vec{\nabla}(cE) \quad , \tag{310}$$

where $D = D(\vec{r},\nu,t)$ is the local diffusion coefficient at frequency ν. This is accomplished by neglecting the $\partial\vec{F}/\partial t$ term in Eq. (309), arguing that for the specific intensity of radiation to be almost isotropic as assumed in writing Eq. (308), the problem must be collision dominated, i.e.,

$$\sigma\vec{F} >> \frac{1}{c}\frac{\partial \vec{F}}{\partial t} \quad . \tag{311}$$

Our Fick's law of diffusion is then

$$\vec{F} = -\frac{1}{3\sigma}\vec{\nabla}(cE) \tag{312}$$

and use of Eq. (312) in Eq. (306) gives the diffusion equation

$$\frac{\partial E}{\partial t} - \vec{\nabla}\cdot\frac{1}{3\sigma}\vec{\nabla}(cE) = c\sigma_a(b - E) \quad . \tag{313}$$

Any diffusion equation such as Eq. (313) has an infinite speed of propagation.

We need supplement either the P-1 (telegrapher's) or diffusion descriptions with initial and boundary conditions. From the initial condition on the equation of transfer, Eq. (86), we compute

$$E(\vec{r},\nu,0) = \frac{1}{c}\int_{4\pi} d\vec{\Omega}\ \Lambda(\vec{r},\nu,\vec{\Omega}) \quad , \tag{314}$$

$$\vec{F}(\vec{r},\nu,0) = \int_{4\pi} d\vec{\Omega}\vec{\Omega}\ \Lambda(\vec{r},\nu,\vec{\Omega}) \quad . \tag{315}$$

These are the appropriate initial conditions, with only Eq. (214) required for the diffusion description.

The boundary conditions are not as straightforward to write down. The structure of the P-1 or diffusion equations requires a single condition between cE and $\vec{F}$ at each boundary point $\vec{r}_s$. It is clear that, because of its simple angular dependence, the Eddington representation of the specific intensity, Eq. (308), cannot satisfy the integro-differential boundary condition, Eq. (85), for an arbitrary incoming distribution Γ. The best one can do is demand that Eq. (85) be satisfied in an integral sense. That is, we use Eq. (308) in Eq. (85), multiply the result by a weight function $w(\vec{\Omega})$, and integrate over all incoming directions. This gives

$$\int_{\vec{n}\cdot\vec{\Omega}<0} d\vec{\Omega}\ w(\vec{\Omega})\left[\frac{1}{4\pi}\left(cE + 3\vec{\Omega}\cdot\vec{F}\right) - \Gamma(\vec{\Omega})\right] = 0 \quad , \tag{316}$$

where $\vec{n}$ is a unit outward normal vector at the surface point $\vec{r}_s$. Equation (316), once $w(\vec{\Omega})$ has been specified, is the required boundary condition.

We consider two choices for $w(\vec{\Omega})$ that are commonly used in practice. The first choice is

$$w(\vec{\Omega}) = \vec{n}\cdot\vec{\Omega} \quad , \tag{317}$$

which yields

$$\frac{1}{4}\, cE(\vec{r}_s) - \frac{1}{2}\, \vec{n}\cdot\vec{F}(\vec{r}_s) = \int_{\vec{n}\cdot\vec{\Omega}<0} d\vec{\Omega}\,|\vec{n}\cdot\vec{\Omega}|\,\Gamma(\vec{r}_s,\vec{\Omega}) \quad . \tag{318}$$

This boundary condition, referred to as the Marshak or Milne condition, has the physical interpretation that the normal component of the incoming flux is the integral quantity conserved in passing from the exact condition, Eq. (85), to the integral condition, Eq. (316). The second choice leads to the so-called Mark boundary condition. To obtain this boundary condition, we represent $\vec{\Omega}$ by a polar angle $\theta \equiv \cos^{-1}(\mu)$, measured with respect to the normal $\vec{n}$, and a corresponding azimuthal angle ϕ. If we set

$$w(\mu,\phi) = \delta(\mu - \mu_o) \quad , \tag{319}$$

and, in addition, choose $\mu_o = -1/\sqrt{3}$, Eq. (316) gives

$$\frac{1}{2}\, cE(\vec{r}_s) - \frac{\sqrt{3}}{2}\, \vec{n}\cdot\vec{F}(\vec{r}_s) = \int_0^{2\pi} d\phi\,\Gamma(r_s,\mu = -1/\sqrt{3}) \quad . \tag{320}$$

Equation (320) has the interpretation that the exact boundary condition is satisfied at a single polar angle point, $\mu = -1/\sqrt{3}$. This particular angle is chosen because of considerations such as the following. Consider time independent transport in a homogeneous, purely absorbing, planar system. According to the equation of transfer, photons incident upon the surface will be absorbed such that, at depth z from the surface, $\exp-(\sigma_a z/\mu)$ represents the probability of survival for photons of polar angle $\theta = \cos^{-1}(\mu)$. On the other hand, the Eddington P-1, or diffusion, approximation gives $\exp-(\sqrt{3}\,\sigma_a z)$ as the survival probability for all photons. Hence, $\mu^2 = 1/3$ can be considered as the average angle associated with the Eddington approximation. Experience indicates that the Marshak-Milne condition, Eq. (318), is generally more accurate than the Mark condition, Eq. (320).

In radiation hydrodynamics problems, the quantities of interest are the radiative energy, flux, and pressure tensor. The equations discussed here give $\vec{E}$ and $\vec{F}$, and $\vec{P}$ follows from

$$\vec{\vec{P}}(\vec{r},t) = \frac{1}{3}\,\vec{\vec{I}}\,E(\vec{r},t) \quad . \tag{321}$$

Although the classical diffusion or Eddington approximation is much simpler than the transport description from which it was derived, it should describe the energy flow due to radiative processes in a semi-quantitative sense. This description will be particularly accurate if the specific intensity of radiation is almost isotropic. Of course, the angular detail of the specific intensity has been lost since the essence of the Eddington approximation is the simple angular dependence assumed in Eq. (308).

B. Asymptotic Diffusion Theory

The critical assumption in reducing the equation of transfer, Eq. (303), to the diffusion description, Eq. (313), is that the specific intensity of radiation is almost isotropic, as expressed quantitatively by Eq. (308). This leads to a diffusion coefficient $D = 1/(3\sigma)$.

One would find other diffusion coefficients if other angular distributions were assumed. We consider the effect of one such distribution here, namely the asymptotic angular distribution of the equation of transfer. This time independent distribution is that found deep within (a few mean free paths from all boundaries) a source free ($b = 0$) homogeneous medium in which photons of different frequencies diffuse independently (the scattering is coherent).

The equation of transfer for this problem is, from Eq. (303)

$$\vec{\Omega}\cdot\vec{\nabla}I + I = \frac{\tilde{\omega}}{4\pi}\int_{4\pi} d\vec{\Omega}'\ I(\vec{\Omega}') \quad , \tag{322}$$

where we have set $\sigma = 1$ (we measure distance in units of the mean free path) and defined

$$\tilde{\omega} = \frac{\sigma_s}{\sigma} \quad . \tag{323}$$

We look for a solution of the form

$$I(\vec{r},\vec{\Omega}) = \psi(\vec{\Omega})e^{\vec{K}\cdot\vec{r}} \quad , \tag{324}$$

where $\psi(\vec{\Omega})$ and $\vec{K}$ are to be determined. Use of Eq. (324) in Eq. (322) gives

$$(\vec{\Omega}\cdot\vec{K} + 1)\psi(\vec{\Omega}) = \frac{\tilde{\omega}}{4\pi}\int_{4\pi} d\vec{\Omega}'\ \psi(\vec{\Omega}') \quad , \tag{325}$$

which gives

$$\psi(\vec{\Omega}) = \frac{\tilde{\omega}}{4\pi(1 + \vec{K}\cdot\vec{\Omega})} \quad , \tag{326}$$

where we have normalized the solution such that

$$\int_{4\pi} d\vec{\Omega}\ \psi(\vec{\Omega}) = 1 \quad . \tag{327}$$

Use of Eq. (326) in Eq. (327) and performing the integration gives the dispersion relationship for $K = |\vec{K}|$ as

$$1 = \frac{\tilde{\omega}}{2K} \ln\left(\frac{1 + K}{1 - K}\right) \quad . \tag{328}$$

We note that only the magnitude of $\vec{K}$ is determined; its direction is arbitrary. Thus an asymptotic solution of Eq. (325) is

$$I(\vec{r},\vec{\Omega}) = \frac{\tilde{\omega}}{4\pi(1 + K\vec{u}\cdot\vec{\Omega})} e^{K\vec{u}\cdot\vec{r}} \quad , \tag{329}$$

where $\vec{u}$ is an arbitrary unit vector. Since the equation of transfer is linear, the general asymptotic solution is obtained as an arbitrary superposition of these solutions for different $\vec{u}$, i.e.,

$$I(\vec{r},\vec{\Omega}) = \int d\vec{u} f(\vec{u}) \frac{\tilde{\omega}}{4\pi(1 + K\vec{u}\cdot\vec{\Omega})} e^{K\vec{u}\cdot\vec{r}} \quad . \tag{330}$$

Integration of Eq. (330) over all $\vec{\Omega}$ gives the energy density E as

$$E(\vec{r}) = \frac{1}{c} \int_{4\pi} d\vec{\Omega}\, I(\vec{r},\vec{\Omega}) = \int d\vec{u}\, f(\vec{u}) e^{K\vec{u}\cdot\vec{r}} \quad , \tag{331}$$

where we have made use of Eq. (327), i.e.,

$$\int_{4\pi} d\vec{\Omega} \frac{\tilde{\omega}}{4\pi(1 + K\vec{u}\cdot\vec{\Omega})} = 1 \quad . \tag{332}$$

Applying the Laplacian operator to Eq. (331) we find

$$\nabla^2(cE) - K^2(cE) = 0 \quad . \tag{333}$$

However, the conservation equation is, by integrating Eq. (322) over all solid angle

$$\vec{\nabla}\cdot\vec{F}(\vec{r}) + (1 - \bar{\omega})cE(\vec{r}) = 0 \quad . \tag{334}$$

A comparison of Eqs. (333) and (334) implies

$$\vec{F} = -D\vec{\nabla}(cE) \quad , \tag{335}$$

with

$$D = \frac{1 - \bar{\omega}}{\sigma K^2} \quad . \tag{336}$$

[Note that in writing Eq. (336) we have introduced σ, which means the gradient in Eq. (335) is in real, not optical, space]. This asymptotic diffusion coefficient has the limiting values

$$\sigma D = 1 \ , \qquad \bar{\omega} = 0 \ ,$$

$$\sigma D = 1/3 \ , \qquad \bar{\omega} = 1 \ ,$$

$$\sigma D \to \frac{4}{(\pi\bar{\omega})^2} \ , \qquad \bar{\omega} \to \infty \ , \tag{337}$$

and varies monotonically with $\bar{\omega}$

This result is used to formulate a diffusion-like approximation to the equation of transfer, Eq. (303), in the following way. The zeroth angular moment of Eq. (303) is just

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{F} = c\sigma_a(b - E) \quad . \tag{338}$$

As stated earlier, we require an additional result relating $\vec{F}$ to E for Eq. (338) to be useful. The assumption in asymptotic diffusion theory is that Eq. (335), derived under quite restrictive circumstances, is generally valid. We then have the diffusion equation

$$\frac{\partial E}{\partial t} - \vec{\nabla}\cdot\left(\frac{1-\tilde{\omega}}{\sigma K^2}\right)\vec{\nabla}(cE) = c\sigma_a(b - E) \quad , \tag{339}$$

with $\tilde{\omega} = \sigma_s/\sigma$ and K given by

$$1 = \frac{\tilde{\omega}}{2K}\, \ell n\left(\frac{1+K}{1-K}\right) \quad . \tag{340}$$

Two variations of this result have been suggested.

Variation #1.

We rewrite the equation of transfer as

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I + \sigma I = \frac{c\sigma\omega}{4\pi} E \quad , \tag{341}$$

where we have defined

$$\omega \equiv \frac{\sigma_a b + \sigma_s E}{\sigma E} \quad . \tag{342}$$

Aside from the time dependence, Eq. (341) looks like the equation we analyzed to obtain our asymptotic results. Thus suggests that ω, rather that $\tilde{\omega}$, be used to compute the diffusion coefficient.

Variation #2

In this case, an attempt is made to account for the time dependence by assuming

$$\frac{\partial I}{\partial t} \approx \frac{1}{4\pi}\frac{\partial(cE)}{\partial t} \quad , \tag{343}$$

and the equation of transfer can then be written

$$\vec{\Omega}\cdot\vec{\nabla}I + \sigma I = \frac{c\sigma\hat{\omega}}{4\pi} E \quad , \tag{344}$$

with $\hat{\omega}$ defined by

$$\hat{\omega} = \frac{1}{\partial E}\left[\sigma_a b + \sigma_s E - \frac{1}{c}\frac{\partial E}{\partial t}\right] \quad . \tag{345}$$

The parameter $\hat{\omega}$ is then used to compute D according to

$$D = \frac{1 - \hat{\omega}}{\sigma K^2} \tag{346}$$

with

$$1 = \frac{\hat{\omega}}{2K} \ln\left(\frac{1 + K}{1 - K}\right) \quad . \tag{347}$$

The boundary condition on asymptotic diffusion theory can be obtained by employing the Marshak-Milne philosophy, namely demanding that the asymptotic angular distribution give the correct incoming flux. Omitting the details, the result is

$$\frac{\bar{\omega}}{4K^2} \ln\left(\frac{1}{1 - K^2}\right) cE(\vec{r}_s) - \frac{1}{2}\, \vec{n}\cdot\vec{F}(\vec{r}_s)$$

$$= \int_{\vec{n}\cdot\vec{\Omega}<0} d\vec{\Omega}\, \left|\vec{n}\cdot\vec{\Omega}\right| I(\vec{r}_s,\vec{\Omega}) \quad , \tag{348}$$

where, as before, $\vec{r}_s$ denotes a surface point and $\vec{n}$ is a unit outward normal vector at the point $\vec{r}_s$. In the two variations of asymptotic diffusion theory just mentioned, one would replace $\bar{\omega}$ in Eq. (348) with either ω or $\hat{\omega}$, and use the corresponding value of K.

As $\bar{\omega}$ approaches unity (pure scattering, or effective pure scattering) all aspects of asymptotic diffusion theory agree with those of the Eddington approximation. In particular, σD goes to 1/3, and the asymptotic boundary condition, Eq. (348), goes to the Eddington condition, Eq. (318). For this reason, it is often said that the Eddington approximation is only strictly valid for almost pure scattering problems. This statement is only true as far as asymptotic solutions in a source free medium are concerned. The accurate statement concerning the Eddington approximation is that it is only strictly valid when the specific intensity of radiation is almost isotropic, without regard to the amount of absorption present. Asymptotic diffusion theory is a strictly proper description of radiative transfer when the specific intensity is in a nearly asymptotic state, as is clear from its derivation.

Finally, we consider the pressure tensor, according to asymptotic analysis. This is obtained by analysis similar to that which led to Eq. (335) for the radiative flux. Omitting the details, the ij component of the pressure tensor is given by

$$P_{ij} = \frac{1}{c}\int_{4\pi} d\vec{\Omega}\Omega_i\Omega_j I(\vec{\Omega})$$

$$= \left(\frac{1}{\sigma^2 K^2}\right)\left(\frac{3\sigma D - 1}{2}\right)\frac{\partial^2 E}{\partial x_i \partial x_j} + \left(\frac{1 - \sigma D}{2}\right)E\,\delta_{ij} \quad . \qquad (349)$$

It can be shown by direct computation from Eq. (349) that

$$\vec{\nabla}\cdot\vec{P} = \sigma D\vec{\nabla}E \qquad (350)$$

in the asymptotic regime, with D being the asymptotic diffusion coefficient.

C. Variable Eddington Factors and Flux Limited Diffusion

One of the difficulties with both the Eddington diffusion description and asymptotic diffusion theory is that they often predict too large a radiative flux. That is, since

$$cE = \int_{4\pi} d\vec{\Omega}\, I(\vec{\Omega}) \quad , \tag{351}$$

$$\vec{F} = \int_{4\pi} d\vec{\Omega}\vec{\Omega}\, I(\vec{\Omega}) \quad , \tag{352}$$

we must have

$$|\vec{F}| \leq cE \quad , \tag{353}$$

with the equality holding only in the streaming limit [when $I(\vec{\Omega})$ is a Dirac delta function in some direction]. In the two diffusion theories just described, one has

$$\vec{F} = -D\vec{\nabla}(cE) \quad , \tag{354}$$

and hence for large gradients, one can obtain a flux that violates Eq. (353).

We discuss various methods that have been suggested to remedy this problem; or more generally, to obtain an approximate description of radiative transfer, either a telegrapher or diffusion description, which is more accurate than Eddington or asymptotic theory.

We begin with the equation of transfer

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I + \sigma I = \frac{c}{4\pi}(\sigma_a b + \sigma_s E) \quad , \tag{355}$$

which has as the first two angular moments

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{F} = c\sigma_a(b - E) \quad , \tag{356}$$

and

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + c\vec{\nabla}\cdot\vec{\vec{P}} + \sigma\vec{F} = 0 \quad . \tag{357}$$

The variable Eddington factor approach to radiative transfer is to define the Eddington tensor $\overleftrightarrow{T}$ as

$$\overleftrightarrow{T} = \overleftrightarrow{P}/E \quad , \tag{358}$$

rewrite Eq. (357) as

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + c\vec{\nabla}\cdot(\overleftrightarrow{T}E) + \sigma\vec{F} = 0 \quad , \tag{359}$$

and postulate an apriori expression for $\overleftrightarrow{T}$ in terms of E, $\vec{F}$, σ_a, σ_s, and b. The vast majority of Eddington tensors have been assumed, or derived, to be of the form

$$\overleftrightarrow{T} = \frac{1-\chi}{2}\overleftrightarrow{I} + \frac{3\chi - 1}{2}\frac{\vec{f}\vec{f}}{f^2} \quad , \tag{360}$$

where $\overleftrightarrow{I}$ is the identity tensor, and

$$\vec{f} = \vec{F}/cE \quad , \tag{361}$$

with $f = |\vec{f}|$. The scalar χ is referred to as the Eddington factor. Equation (360) follows uniquely if one assumes that the only vector that $\overleftrightarrow{T}$ depends upon is $\vec{f}$. An equivalent assumption is that the angular distribution is azimuthally symmetric about the direction defined by $\vec{f}$, and χ is then given by

$$\chi = \frac{\int_{-1}^{1} d\mu\mu^2 I(\vec{\Omega})}{\int_{-1}^{1} d\mu I(\vec{\Omega})} \quad , \tag{362}$$

where $\mu = \vec{\Omega}\cdot\vec{f}/f$. The Eddington approximation previously discussed corresponds to $\chi = 1/3$. We note from its definition that the trace of $\vec{T}$ must be unity, and Eq. (360) has this property.

One also has the inequalities

$$0 \leq f \leq 1 \quad , \tag{363}$$

$$f^2 \leq \chi \leq 1 \quad . \tag{364}$$

Equation (363) is just a rewriting of Eq. (353), and Eq. (364) is just an application of the Schwartz inequality. The prescriptions that have been suggested, or derived, for χ are generally of the functional form

$$\chi = \chi(f,\omega) \quad . \tag{365}$$

That is, the Eddington factor depends upon the magnitude of the dimensionless flux $\vec{f}$ and the effective single scatter albedo ω defined as

$$\omega = \frac{\sigma_a b + \sigma_s E}{\sigma E} \quad . \tag{366}$$

If $I(\vec{\Omega})$ in Eq. (362) is isotropic, we obtain $\chi = 1/3$, the Eddington result.

Since isotropic intensity corresponds to $f = 0$, we expect all reasonable functions χ to have the limiting form

$$\chi(0,\omega) = 1/3 \quad . \tag{367}$$

At the other extreme, if $I(\vec{\Omega})$ in Eq. (362) is a Dirac delta function $\delta(1 - \mu)$ or $\delta(1 + \mu)$, which corresponds to unidirectional streaming, we obtain $\chi = 1$. For this angular distribution, $f = 1$, and hence we should have

$$\chi(1,\omega) = 1 \quad . \tag{368}$$

Since χ and (to a lesser extent) f are both measures of the anisotropy of the specific intensity of radiation, one would qualitatively expect that χ would vary monotonically between these two limits.

The flux limiting approach to radiative transfer is to replace Eq. (357), the first moment, with a Fick's law of diffusion

$$\vec{F} = -\frac{D}{\sigma}\vec{\nabla}(cE) \tag{369}$$

where the (dimensionless) diffusion coefficient is postulated, or derived, as a functional of E, σ_S, σ_S, and b. The idea here is to choose a functional form of D such that the resulting diffusion theory is fully flux limited, i.e.,

$$\left|\vec{F}\right| \le cE \quad , \tag{370}$$

as stated in Eq. (363). In general, the suggested forms for D have depended upon the dimensionless gradient

$$X = \frac{\left|\vec{\nabla}E\right|}{\sigma E} \quad , \tag{371}$$

and the effective albedo ω, or equivalently,

$$D = D(R,\omega) \quad , \tag{372}$$

where

$$R = \frac{X}{\omega} = \frac{\left|\vec{\nabla}E\right|}{\sigma\omega E} \quad . \tag{373}$$

On qualitative physical grounds, one would expect any reasonable prescription for the diffusion coefficient to reproduce classical diffusion theory in the limit of near thermodynamic equilibrium ($R = 0$, $\omega = 1$), i.e.,

$$D(0,1) = 1/3 \quad . \tag{374}$$

In addition, one must have

$$D(R,\omega) \xrightarrow[R\to\infty]{} 0 \quad , \tag{375}$$

for Fick's law, Eq. (369), to yield a finite flux in the limit of infinite gradients. One would also expect D to be a monotonically decreasing function of R to properly maintain flux limiting.

We note that Fick's law, Eq. (369), could be generalized to involve a tensor diffusion coefficient of the form

$$c\vec{\vec{D}}\cdot\vec{\nabla}E + \sigma\vec{F} = 0 \quad . \tag{376}$$

This complexity is probably unwarranted since any Fick's law is approximate in any event.

Two Examples

Before proceeding more generally, we give two examples, one assumed (as did the originator) and one derived (as did the originator) of flux limited diffusion coefficients and Eddington factors.

Example #1

J. Wilson of Lawrence Livermore Laboratory proposed a diffusion coefficient of the form

$$D = \frac{1}{\omega(3 + R)} \quad , \tag{377}$$

which gives a Fick's law of the form

$$\vec{F} = -\left[\frac{1}{3\omega\sigma + |\vec{\nabla}E|/E}\right] \vec{\nabla}(cE) \quad . \tag{378}$$

We note the three properties:

(1) In near thermodynamic equilibrium ($R \approx 0$, $\omega \approx 1$), this reduces to the Eddington result

$$\vec{F} = -\frac{1}{3\sigma} \vec{\nabla}(cE) \quad . \tag{379}$$

(2) For infinite gradients, we obtain the streaming result

$$|\vec{F}| = cE \quad . \tag{380}$$

(3) D decreases monotonically with R.

This model was proposed in an <u>adhoc</u> manner, as a simple functional form having these three properties.

<u>Example #2</u>

By contrast, Levermore of LLL used the ideas of the Chapman-Enskog theory of gases to <u>derive</u> a flux limited diffusion coefficient and a corresponding Eddington factor. We give here a simplified derivation. We begin with the equation of transfer

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I + \sigma I = \frac{c}{4\pi}(\sigma_a b + \sigma_s E) \quad , \tag{381}$$

and its zeroth moment

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{F} = c\sigma_a(b - E) \quad . \tag{382}$$

We introduce the normalized specific intensity $\psi(\vec{r},\nu,\vec{\Omega},t)$ by the equation

$$I = cE\psi \ , \qquad (383)$$

where ψ is normalized to

$$\int_{4\pi} d\vec{\Omega}\ \psi(\vec{\Omega}) = 1 \ . \qquad (384)$$

The function ψ is known in two limiting cases. In the Eddington (isotropic) limit we have

$$\psi = \frac{1}{4\pi}\left[1 - \vec{\Omega}\cdot\vec{\nabla}E/\sigma E\right] \ , \qquad (385)$$

where $|\vec{\nabla}E|/\sigma E$ is assumed to be small. In the streaming limit, we have

$$\psi = \delta_2(\vec{\Omega} - \vec{\Omega}_s) \ , \qquad (386)$$

where $\delta_2(\vec{\Omega} - \vec{\Omega}_s)$ is the angular Dirac delta function indicating streaming in the direction $\vec{\Omega}_s$. Use of Eq. (383) in Eqs. (381) and (382) gives

$$\frac{1}{c}\frac{\partial(E\psi)}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}(E\psi) + \sigma E\psi = \frac{1}{4\pi}\left(\sigma_a b + \sigma_s E\right) \ , \qquad (387)$$

$$\frac{1}{c}\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(E\vec{f}) = \sigma_a(b - E) \ . \qquad (388)$$

Here $\vec{f}$ is the normalized radiative flux defined by

$$\vec{F} = cE\vec{f} \ , \qquad (389)$$

or equivalently

$$\vec{f} = \int_{4\pi} d\vec{\Omega}\vec{\Omega}\ \psi(\vec{\Omega}) \quad . \tag{390}$$

We now use Eq. (388) to eliminate $\partial E/\partial t$ in Eq. (387). This gives

$$\left(\frac{1}{c}\frac{\partial\psi}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}\psi\right)E + \left(\vec{\Omega}\cdot\vec{\nabla}E - \vec{f}\cdot\vec{\nabla}E - E\vec{\nabla}\cdot\vec{f} + \sigma_s E + \sigma_a b\right)\psi$$

$$= \frac{1}{4\pi}(\sigma_a b + \sigma_s E) \quad . \tag{391}$$

Equation (391) is exact. To proceed, we make the assumption that the normalized intensity is a slowly varying function of space and time. Specifically, we set

$$\frac{1}{c}\frac{\partial\psi}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}\ \psi = 0 \quad . \tag{392}$$

The justification for Eq. (392) is that it is true in the two limiting case just discussed, and hopefully introduces a small error in intermediate situations. Use of Eq. (392) in Eq. (391) gives

$$(\vec{\Omega}\cdot\vec{\nabla}E - \vec{f}\cdot\vec{\nabla}E + \sigma\omega E)\psi = \frac{\sigma\omega E}{4\pi} \quad , \tag{393}$$

where ω is the effective albedo given by

$$\omega = \frac{\sigma_a b + \sigma_s E}{\sigma E} \quad . \tag{394}$$

In obtaining Eq. (393) we have used

$$\vec{\nabla}\cdot\vec{f} = 0 \quad , \tag{395}$$

which follows from integration of Eq. (392) over all solid angle and Eq. (384). If we define the vector $\vec{R}$ as

$$\vec{R} = -\frac{\vec{\nabla}E}{\sigma\omega E} , \tag{396}$$

we can solve Eq. (393) for ψ as

$$\psi = \frac{1}{4\pi}\left[\frac{1}{1 + \vec{f}\cdot\vec{R} - \vec{\Omega}\cdot\vec{R}}\right] . \tag{397}$$

From Eqs. (390) and (397), it is clear that the vectors $\vec{f}$ and $\vec{R}$ are in the same direction, and we write

$$\vec{f} = \lambda(R)\vec{R} , \tag{398}$$

where $R = |\vec{R}|$. The proportionality function $\lambda(R)$ between $\vec{f}$ and $\vec{R}$ follows by demanding that ψ be properly normalized. We find, using Eq. (397) in Eq. (384)

$$1 = \frac{1}{4\pi}\int_{4\pi} d\vec{\Omega}\left[\frac{1}{1 + \lambda R^2 - \vec{\Omega}\cdot\vec{R}}\right] = \frac{1}{R}\tanh^{-1}\left(\frac{R}{1 + \lambda R^2}\right) . \tag{399}$$

Solving this for $\lambda(R)$, we find

$$\lambda(R) = \frac{1}{R}\left(\coth R - \frac{1}{R}\right) . \tag{400}$$

Alternately, one could obtain this same result by using Eq. (397) in the defining function for $\vec{f}$, namely Eq. (390). Use of Eqs. (398) and (400) in Eq. (397) gives the angular distribution in terms of the vector $\vec{R}$ as

$$\psi(\vec{\Omega}) = \frac{1}{4\pi}\left[\frac{1}{R\coth R - \vec{\Omega}\cdot\vec{R}}\right] . \tag{401}$$

It is easily seen that in the two limits previously discussed [see Eq. (385) and (386)], which here corresponds to $R \to 0$ and $R \to \infty$, Eq. (401) gives the proper angular distributions.

To obtain a Fick's law, we combine Eqs. (389), (396), (398), and (400) to obtain

$$\vec{F} = -\frac{D}{\sigma}\vec{\nabla}(cE) \quad , \tag{402}$$

where the dimensionless diffusion coefficient D is given by

$$D = \frac{1}{\omega R}\left(\coth R - \frac{1}{R}\right) \quad . \tag{403}$$

We see this Fick's law has the properties:

(1) In near thermodynamic equilibrium ($R \approx 0$, $\omega \approx 1$)

$$D \to 1/3 \quad . \tag{404}$$

(2) For any value of R and ω

$$|\vec{F}| < cE \quad , \tag{405}$$

and, in particular, as $R \to \infty$

$$|\vec{F}| = cE \quad . \tag{406}$$

(3) D is monotonically decreasing with R.

To obtain the corresponding Eddington factor χ, one uses the angular distribution, Eq. (401), in the expression for χ given by Eq. (362). Performing the integrations, one finds

$$\chi = \coth R\left[\coth R - \frac{1}{R}\right] \quad . \tag{407}$$

If one wishes χ as a function of f, rather than R as in Eq. (407), one uses Eqs. (398) and (400) to obtain

$$f = \coth R - 1/R \quad . \tag{408}$$

Elimination of R between Eqs. (407) and (408) gives χ as a function of f. It is easily shown that $\chi(f)$ has the three properties:

(1) $\chi(0) = 1/3$,

(2) $\chi(1) = 1$,

(3) $\chi(f)$ is monotonically increasing with f. (409)

A Relationship Between Eddington Factors and Flux Limiters

Returning now to more general considerations, we derive an (approximate) relationship between χ and D in a fairly general way. We begin with the first moment equation given by Eq. (359), i.e.,

$$\frac{1}{c}\frac{\partial \vec{F}}{\partial t} + c\vec{\nabla}\cdot(\vec{\vec{T}}E) + \sigma\vec{F} = 0 \tag{410}$$

with $\vec{\vec{T}}$ given by

$$\vec{\vec{T}} = \frac{1-\chi}{2}\vec{\vec{I}} + \frac{3\chi - 1}{2}\frac{\vec{f}\vec{f}}{f^2} \quad . \tag{411}$$

We eliminate the $\partial\vec{F}/\partial t$ term in Eq. (410) by writing

$$\frac{\partial \vec{F}}{\partial t} = \frac{\vec{F}}{E}\frac{\partial E}{\partial t} \quad . \tag{412}$$

The justification for Eq. (412) is that is correct in the isotropic ($f = 0$) and streaming ($f = 1$) limits, and hopefully is reasonably accurate in general. We then have

$$\frac{1}{c}\,\frac{\vec{F}}{E}\,\frac{\partial E}{\partial t} + c\vec{\nabla}\cdot(\overleftrightarrow{T}E) + \sigma\vec{F} = 0 \quad . \tag{413}$$

We use the conservation equation, i.e.,

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\vec{F} = c\sigma_a(b - E) \quad , \tag{414}$$

to eliminate $\partial E/\partial t$ in Eq. (413). This gives

$$\vec{F} = -\frac{1}{\sigma\omega}\left[1 - \frac{\vec{\nabla}\cdot\vec{F}}{\sigma\omega cE}\right]^{-1} c\vec{\nabla}\cdot(\overleftrightarrow{T}E) \quad , \tag{415}$$

with ω once again given by

$$\omega = \frac{\sigma_a b + \sigma_s E}{\sigma E} \quad . \tag{416}$$

To obtain a Fick's law of diffusion of the form

$$\vec{F} = -\frac{D}{\sigma}\,\vec{\nabla}(cE) \quad , \tag{417}$$

it is necessary to assume that the Eddington tensor $\overleftrightarrow{T}$ is slowly varying in space so that one can write

$$\vec{\nabla}\cdot(\overleftrightarrow{T}E) = \overleftrightarrow{T}\cdot\vec{\nabla}E \quad . \tag{418}$$

From Eqs. (411), (415), and (418) it is easily shown that the vectors $\vec{F}$ and $\vec{\nabla}E$ are proportional to one another. Then, using Eq. (411) in Eq. (418) we find

$$\vec{T}\cdot\vec{\nabla}E = \chi\vec{\nabla}E \quad , \tag{419}$$

and hence Eq. (415) becomes

$$\vec{F} = -\frac{\chi}{\sigma\omega}\left[1 - \frac{\vec{\nabla}\cdot\vec{F}}{\sigma\omega cE}\right]^{-1}\vec{\nabla}(cE) \quad . \tag{420}$$

Equation (420) is a Fick's law of diffusion with the diffusion coefficient D, in dimensionless form, given by

$$D = \frac{\chi}{\omega}\left[1 - \frac{\vec{\nabla}\cdot\vec{F}}{\sigma\omega cE}\right]^{-1} \quad . \tag{421}$$

Now, by assumption, χ depends upon f and ω, and we want D to depend upon only R and ω. This means we must eliminte the unwanted functional dependence $\vec{\nabla}\cdot\vec{F}$ from Eq. (421). We do this by writing

$$\vec{F} = \vec{f}cE \quad , \tag{422}$$

and assuming $\vec{f}$ is slowly varying in space. We then have

$$\vec{\nabla}\cdot\vec{F} = c\vec{f}\cdot\vec{\nabla}E = -cf\left|\vec{\nabla}E\right| \quad , \tag{423}$$

with the last equality following from the fact that the vector $\vec{f}$ and $\vec{\nabla}E$ are in opposite directions [see Eq. (417)]. We the obtain from Eq. (421)

$$D = \frac{\chi}{\omega(1 + fR)} \tag{424}$$

where

$$R = \frac{\left|\vec{\nabla}E\right|}{\sigma\omega E} \quad . \tag{425}$$

We obtain a second relationship by dividing Fick's law, i.e.,

$$\vec{F} = -\frac{D}{\sigma}\vec{\nabla}(cE) \quad , \tag{426}$$

by cE and taking the absolute value. This gives

$$f = D\omega R \quad . \tag{427}$$

Given $\chi = \chi(f,\omega)$, elimination of f between Eqs. (424) and (427) gives $D = D(R,\omega)$. Conversely, given $D = D(R,\omega)$, elimination of R between Eqs. (424) and (427) gives $\chi = \chi(f,\omega)$.

A Comparison of Various Flux Limiters and Eddington Factors

We examine the properties of certain flux limited diffusion coefficients and Eddington factors that have been proposed and are currently in use. In this examination we pay attention to the inequalities given by Eqs. (363) and (364); the limiting expressions given by Eqs. (367), (368), (374), and (375); and the monotonicity properties discussed earlier. We consider the various prescriptions in roughly chronological order of their introduction.

1. The Eddington Approximation

The classical Eddington approximation corresponds to

$$D = 1/3 \quad , \tag{428}$$

which clearly violates the large R limit given by Eq. (375). This is just a manifestation that classical diffusion theory is not flux limited. Equations (424) and (427) give as the corresponding Eddington factor

$$\chi = \frac{\omega}{3} + f^2 \quad . \tag{429}$$

Except for $\omega = 0$, Eq. (429) violates the inequality $\chi < 1$; in particular, Eq. (368) is not satisfied. Hence we conclude, as is well known, that the classical Eddington approximation is limited in its region of validity.

2. The Wilson (Sum) Flux Limiter

As previously mentioned, J. Wilson of LLL suggested the form

$$D = \frac{1}{\omega(3 + R)} \quad , \tag{430}$$

as a means of introducing flux limiting into diffusion theory. This form has all the desirable properties for a diffusion coefficient as previously discussed. From Eqs. (424) and (427), we find the corresponding Eddington factor is given by

$$\chi = \frac{1}{3}(1 - f + 3f^2) \quad . \tag{431}$$

Although Eq. (431) gives the correct limiting behavior at $f = 0$ and $f = 1$, it is not a monotonic function of f. χ, as given by Eq. (431), has a minimum value of 11/36 at $f = 1/6$. Hence, from an examination of the Eddington factor corresponding to the diffusion coefficient, we conclude that the diffusion coefficient itself may be less than satisfactory. In particular, one can conjecture that this diffusion coefficient probably introduces too much flux limiting, thereby underestimating the flux.

3. The Wilson (Maximum) Flux Limiter

One can avoid the minimum in χ just discussed by replacing Eq. (430) by

$$D = \frac{1}{\omega[\max(3,R)]} \quad . \tag{432}$$

This gives for the corresponding Eddington factor

$$\chi = \frac{1}{3} + f^2 \ , \quad R \lesssim 3 \quad . \tag{433}$$

For $R > 3$, Eqs. (424) and (427) cannot be solved for χ. In particular, Eq. (427) simply gives $f = 1$. If we interpret Eq. (433) to hold for all f in the physical range $0 \leq f \leq 1$, we see the inequality $\chi < 1$ is violated for $f^2 > 2/3$. Since χ is too large for f near unity, one can conjecture that the diffusion coefficient may in general overestimate the flux, i.e., not give enough flux limiting.

4. The Wilson (Fit) Flux Limiter

By fitting to certain transport calculations, Wilson has suggested a diffusion coefficient given by

$$D = \frac{1}{3 + \omega R[1 + 3\exp - (\omega R/2)]} \ , \tag{434}$$

which has all of the desired properties discussed earlier. For this complex functional form, one cannot analytically solve Eqs. (424) and (427) for $\chi(f,\omega)$. However, in the limits of small and large R, one finds the results

$$\chi = \frac{\omega}{3}(1 - 4f) + O(f^2) \ , \quad f \ll 1 \ , \tag{435}$$

$$\chi = \frac{1}{3}[\omega(1-f) + 3f^2] + O(\exp - [\frac{3}{2(1-f)}]), \ (1-f) \ll 1 \ . \tag{436}$$

We see that $\chi = 1$ at $f = 1$ in accord with Eq. (368). However, only for $\omega = 1$ do we recover the $f = 0$ limit given by Eq. (367). We also note that

$$\left.\frac{\partial \chi}{\partial f}\right|_{f=0} = -\frac{4\omega}{3} < 0 \quad , \tag{437}$$

which implies that the curve χ vs. f is not monotonic but goes through a minimum at some point. Once again, this suggests that the corresponding diffusion coefficient, Eq. (435), may underestimate the flux.

5. Asymptotic Diffusion Theory

As previously discussed, in asymptotic diffusion theory the diffusion coefficient is given as a function of ω (or $\hat{\omega}$ or $\tilde{\omega}$) alone according to

$$D = \frac{1 - \omega}{K^2} \quad , \tag{438}$$

where K satisfies the transcendental equation

$$\frac{2K}{\omega} = \ln \left(\frac{1 + K}{1 - K}\right) \quad . \tag{439}$$

This diffusion coefficient has the limiting forms

$$D(\omega = 0) = 1 \; ; \quad D(\omega = 1) = 1/3 \quad . \tag{440}$$

We see that Eq. (438) gives the proper thermodynamic limit, Eq. (374), but clearly violates the large R limit, Eq. (375). The corresponding Eddington factor is found from Eqs. (424) and (427) to be

$$\chi = \frac{\omega(1 - \omega)}{K^2} + f^2 \quad . \tag{441}$$

This expression, for a general ω, satisfies neither the $f = 0$ limit, Eq. (367), nor the $f = 1$ limit, Eq. (368). Thus it appears that asymptotic diffusion theory is of limited validity.

6. The Winslow Flux Limiter

To improve upon asymptotic diffusion theory, A. Winslow (LLL) suggested a flux limiter of the form

$$D = \frac{D_A}{\max[1, \omega \ell R]} \quad , \tag{442}$$

where D_A is the asymptotic diffusion coefficient given by Eq. (438) and $\ell = \ell(\omega)$ is the linear extrapolation for the Milne problem. Limiting values are

$$\ell(\omega = 0) = 1 \; ; \quad \ell(\omega = 1) = 0.7104 \quad . \tag{443}$$

This function has the desired properties for a diffusion coefficient; in particular, this D vanishes for large R, as contrasted with the pure asymptotic diffusion form given by Eq. (438). From Eqs. (424) and (427) we find

$$\chi = \frac{\omega(1 - \omega)}{K^2} + f^2 \; , \quad \omega \ell R \leq 1 \quad . \tag{444}$$

For $\omega \ell R > 1$, Eqs. (424) and (427) cannot be solved for χ. Specifically, Eq. (427) simply gives

$$f = \frac{D_A}{\ell} = \frac{1 - \omega}{K^2 \ell} \; , \quad \omega \ell R > 1 \quad . \tag{445}$$

If we interpret Eq. (444) to hold for all f in the range $0 \leq f \leq 1$, we see that the inequality $\chi \leq 1$ is violated for large f. We also note that the $f = 0$ limit, Eq. (367), is not satisfied except for $\omega = 1$. Thus the form for D given by Eq. (442), while introducing flux limiting into asymptotic diffusion theory, gives a somewhat non-physical Eddington factor.

7. Kershaw's Eddington Factor

Applying the theory of moments, D. Kershaw (LLL) developed a series of inequalities which the angular moments of the specific intensity $I(\vec{\Omega})$ must satisfy. Examples of these inequalities are given by Eqs. (363) and (364). On the basis of these inequalities, he suggested an Eddington factor given by

$$\chi = \frac{1}{3}\left(1 + 2f^2\right) \quad , \tag{446}$$

which has all the proper behavior previously discussed. The corresponding diffusion coefficient is, from Eqs. (424) and (427),

$$D = \frac{\sqrt{9 + 4R^2} - 3}{2\omega R^2} \quad . \tag{447}$$

This diffusion coefficient is monotonically decreasing as a function of R, and gives the correct behavior for small and large R [see Eqs. (374) and (375)]. Thus, Kershaw's prescription gives both an Eddington factor and diffusion coefficient with all of the qualitatively correct properties as discussed earlier.

8. Minerbo's (Statistical) Eddington Factor

Treating photons as a statistical ensemble with E and $\vec{F}$ prescribed as constraints, G. Minerbo, LANL, computed the most likely angular distribution for the specific intensity. From this distribution he deduced an Eddington factor as a function of f given parametrically by

$$\chi = 1 - \frac{2f}{C} \quad , \tag{448}$$

$$f = \coth C - \frac{1}{C} \quad . \tag{449}$$

Eliminating C between these two equations gives $\chi = \chi(f)$. Setting $C = 0$ and $C = \infty$, respectively, in Eqs. (448) and (449) gives the proper limiting values

$$\chi(f = 0) = 1/3 \ ; \qquad \chi(f = 1) = 1 \quad . \tag{450}$$

Further, this χ increases monotonically between these two limits. The corresponding diffusion coefficient can be written as

$$D(R,\omega) = \frac{\lambda(R)}{\omega} \quad , \tag{451}$$

where the function λ depends only upon R. This functional dependence is obtained by eliminating C between the two equations

$$(\lambda R)^2 + \left(\frac{1}{R} + \frac{2}{C}\right) (\lambda R) - 1 = 0 \quad , \tag{452}$$

$$\lambda R = \coth C - \frac{1}{C} \quad . \tag{453}$$

Setting $C = 0$ and $C = \infty$, respectively, Eqs. (452) and (453) gives the limiting behavior

$$\lambda(R = 0) = 1/3 \ ; \qquad \lambda \xrightarrow[R \to \infty]{} 1/R \quad . \tag{454}$$

The function $\lambda(R)$ decreases monotonically as R increases. Thus the Minerbo treatment also gives both an Eddington factor and diffusion coefficient that are qualitatively correct.

9. Minerbo's (Linear) Eddington Factor

As an approximation to the angular distribution resulting from his statistical arguments, Minerbo considered a linear (in μ) angular distribution for the specific intensity of radiation, with the constraint that this distribution be non-negative. This gave an Eddington factor

$$\chi = 1/3 \quad , \qquad 0 \leq f \leq 1/3 \quad , \tag{455a}$$

$$\chi = \frac{1}{2} - f + \frac{3}{2} f^2 \quad , \qquad 1/3 \leq f \leq 1 \quad , \tag{455b}$$

which has the correct monotonic behavior and limiting values. The corresponding diffusion coefficient can be written in the form given by Eq. (451) with

$$\lambda(R) = \frac{\sqrt{9 + 12R^2} - 3}{6R^2} \quad , \qquad 0 \leq R \leq 3/2 \quad , \tag{456a}$$

$$\lambda(R) = \frac{(R + 1) - \sqrt{2R + 1}}{R^2} \quad , \qquad 3/2 \leq R < \infty \quad . \tag{456b}$$

It is clear from Eq. (456) that $\lambda(R)$ is a monotonically decreasing function of R, with limiting behavior

$$\lambda(R = 0) = 1/3 \ ; \qquad \lambda \xrightarrow[R \to \infty]{} 1/R \quad . \tag{457}$$

Thus, Minerbo's linear treatment gives qualitatively correct results for both the Eddington factor and the diffusion coefficient, although the Eddington factor has a somewhat unrealistic flat behavior for $f \leq 1/3$.

10. Levermore's (Chapman-Enskog) Diffusion Coefficient

As previously described, Levermore obtained a diffusion coefficient given by Eq. (451) with

$$\lambda(R) = \frac{1}{R}\left(\coth R - \frac{1}{R}\right) , \tag{458}$$

which is properly monotonically decreasing from a value of 1/3 at R = 0 to a 1/R behavior for large R. The Eddington factor associated with Eq. (458) follows from Eqs. (424), (427), and (451) as

$$\chi = \coth R\left(\coth R - \frac{1}{R}\right) , \tag{459}$$

$$f = \coth R - \frac{1}{R} . \tag{460}$$

Elimination of R between Eqs. (459) and (460) gives χ as a function of f. We note that Eq. (459) is the same result one obtains by computing the Eddington factor directly from the angular distribution associated with the Levermore theory [see Eq. (407)]. The above functional form for χ has the limiting values

$$\chi(f = 0) = 1/3 \; ; \quad \chi(f = 1) = 1 , \tag{461}$$

and varies monotonically between these two limits. Thus we see that the Levermore treatment gives proper behavior for both D and χ.

11. Levermore's (Lorentz) Eddington Factor

In a separate approach, Levermore applied a Lorentz transformation to the equation of transfer, transforming to a frame in which the radiative flux is zero. In this frame he assumed that the Eddington factor is 1/3. Transforming back to the original frame, he obtained

$$\chi = \frac{1 + 3\beta^2}{3 + \beta^2} , \tag{462}$$

$$f = \frac{4\beta}{3 + \beta^2} , \tag{463}$$

where $\beta = |\vec{v}|/c$, with c the speed of light and $\vec{v}$ the velocity of the transformed frame with respect to the original frame. Eliminating β between these two equations gives χ explicitly as a function of f, i.e.,

$$\chi = \frac{1}{3} \left(5 - 2\sqrt{4 - 3f^2}\right) . \tag{464}$$

We note that this functional form has all of the qualitative properties that an Eddington factor should have. We find that the corresponding diffusion coefficient is given by Eq. (451), with $\lambda(R)$ determined by eliminating β between the two equations

$$\lambda = 3\left(\frac{1 - \beta^2}{3 + \beta^2}\right)^2 , \tag{465}$$

$$R = \frac{4\beta(3 + \beta^2)}{3(1 - \beta^2)^2} . \tag{466}$$

In the limits of $\beta = 0$ and $\beta = 1$, one obtains

$$\lambda(R = 0) = 1/3 \; ; \quad \lambda \xrightarrow[R\to\infty]{} 1/R , \tag{467}$$

and the behavior between these two limits is monotonic. Hence, this treatment also gives qualitatively correct behavior for both the Eddington factor and the diffusion coefficient.

We can summarize these results as follows: Of the eleven different treatments of Eddington factors and diffusion coefficients, five were shown to be qualitatively correct. These five treatments are:

1. Kershaw's
2. Minerbo's (statistical)
3. Minerbo's (linear)
4. Levermore's (Chapman-Enskog)
5. Levermore's (Lorentz)

In all five cases, the diffusion coefficient can be written

$$D(R,\omega) = \frac{\lambda(R)}{\omega} , \tag{468}$$

and Figure 1 plots $\lambda(R)$ for these five different approaches. We see that all five curves have a very similar behavior, decreasing monotonically between the limits common to all five curves

$$\lambda(0) = 1/3 ; \quad \lambda(R) \xrightarrow[R\to\infty]{} \frac{1}{R} . \tag{469}$$

These five curves also have the common characteristic

$$\left.\frac{\partial\lambda(R)}{\partial R}\right|_{R=0} = 0 . \tag{470}$$

Similarly, Figure 2 plots the Eddington factor χ, which in all five cases is a function of f alone. Again we see that all of the curves behave similarly, increasing monotonically as f increases, and sharing the common characteristics

$$\chi(0) = 1/3 \; ; \quad \left.\frac{\partial \chi(f)}{\partial f}\right|_{f=0} = 0 \; ; \quad \chi(1) = 1 \; . \qquad (471)$$

We note that the spread in the Eddington factor curves is somewhat greater than in the diffusion coefficient (or λ) curves, indicating that D is a weaker functional of χ than χ is of D.

It is probably difficult, if not impossible, to single out any of these five treatments as "best". Which of the five will perform the best is undoubtedly problem dependent. On the other hand, since the curves are all quite similar, it probably is relatively unimportant which of the five treatments is adopted. One can conjecture that all will give comparable accuracy when applied to a variety of problems, although the linear treatment of Minerbo could perhaps be expected to be somewhat less accurate because of the unrealistic flat behavior of χ for $f \leq 1/3$. One can also conjecture that the other six treatments we have discussed are probably inferior over a wide range of problems in that they each display at least one qualitatively incorrect characteristic.

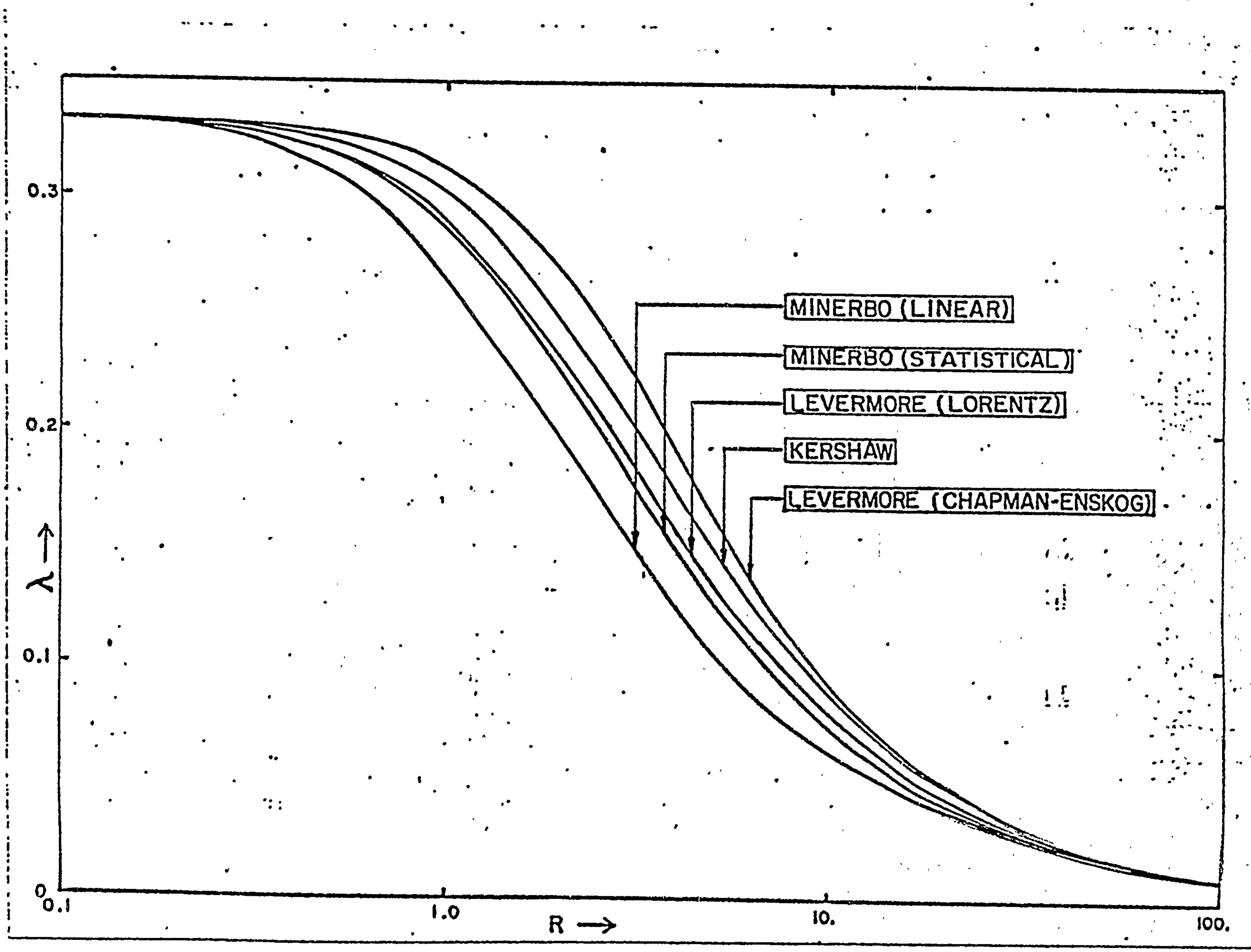

MINERBO (LINEAR)
MINERBO (STATISTICAL)
LEVERMORE (LORENTZ)
KERSHAW
LEVERMORE (CHAPMAN-ENSKOG)
0.3
0.2
0.1
0
λ
0.1
1.0
10.
100.
R

FIGURE 1

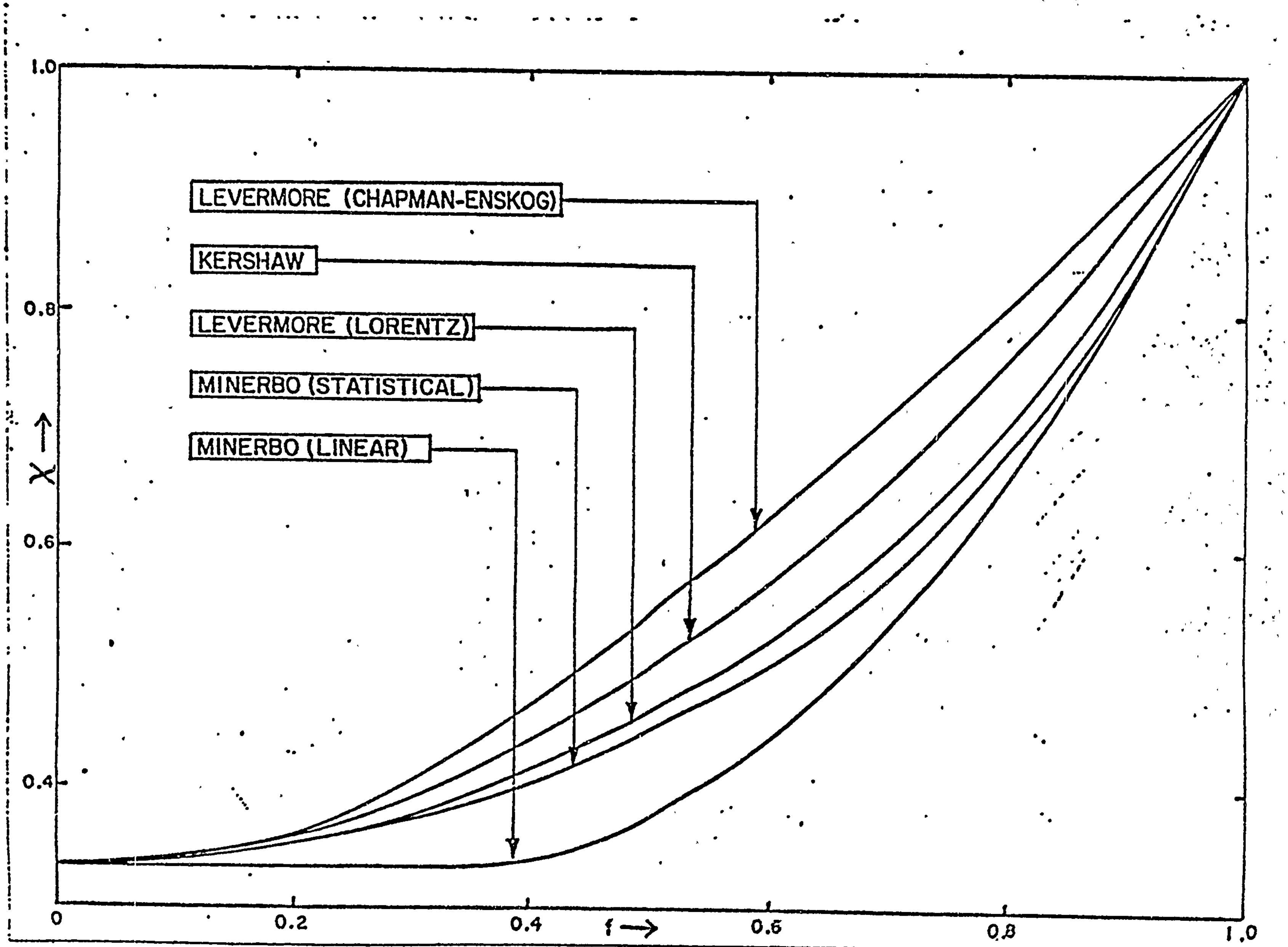

LEVERMORE (CHAPMAN-ENSKOG)
KERSHAW
LEVERMORE (LORENTZ)
MINERBO (STATISTICAL)
MINERBO (LINEAR)
1.0
0.8
0.6
0.4
χ
0
0.2
0.4
f
0.6
0.8
1.0

FIGURE 2

D. Equilibrium Diffusion Theory

The simplest treatment of radiative transfer, and radiation-hydrodynamics, is what is generally referred to as equilibrium diffusion theory. This is a further simplification of the Eddington approximation discussed earlier, and really constitutes an approximate solution of the Eddington equations. That is, given the material temperature distribution within a specific system, the equilibrium diffusion approximation provides an explicit expression giving the specific intensity as a function of all its variables, namely space, frequency, angle, and time. In particular, it gives the energy density, radiative flux, and pressure tensor as functionals of the material temperature, and hence as implicit functions of space and time, as required in the equations of hydrodynamics when a radiation field is present. Although equilibrium diffusion theory corresponds to a very low order approximation in both frequency and angle (as well as space and time) it is, because of its simplicity, a widely used calculational scheme in many radiation-hydrodynamic problems. Surprisingly enough, in view of all the approximations made, it turns out to be a reasonably accurate description for many problems, giving gross features of the radiation flow correctly, in a qualitative and even a semi-quantitative sense.

We begin with the Eddington moment equations, namely [see Eqs. (306) and (312)]

$$\frac{\partial E(\nu)}{\partial t} + \vec{\nabla}\cdot\vec{F}(\nu) = c\sigma_a(\nu)[b(\nu,T) - E(\nu)] \quad , \tag{472}$$

$$\vec{F}(\nu) = -\frac{1}{3\sigma(\nu)}\vec{\nabla}[cE(\nu)] \quad , \tag{473}$$

where $\vec{r}$ and t dependences of all quantities is understood. The underlying assumptions in these equations are: (1) the specific intensity is almost isotropic as expressed quantitatively by Eq. (308), i.e.,

$$I(\nu,\vec{\Omega}) = \frac{1}{4\pi}\left[cE(\nu) + 3\vec{\Omega}\cdot\vec{F}(\nu)\right] \quad , \tag{474}$$

and (2) the problem is collision dominated so that $\partial\vec{F}/\partial t$ can be neglected compared to $c\sigma\vec{F}$. We assume that the left hand side of Eq. (472) is sufficiently small so that it can be neglected. The Eq. (472) becomes

$$c\sigma_a(\nu)\left[b(\nu,T) - E(\nu)\right] = 0 \quad , \tag{475}$$

which implies that the radiation energy density is locally Planckian at the local material temperature, i.e.,

$$E(\vec{r},\nu,t) = b(\nu,T) = \frac{4\pi}{c} B(\nu,T) \quad . \tag{476}$$

Use of this result in Eq. (473) gives

$$F(\vec{r},\nu,t) = -\frac{4\pi}{3\sigma(\vec{r},\nu,t)} \vec{\nabla}B(\nu,T) \quad , \tag{477}$$

where $T = T(\vec{r},t)$. The gradient operator acts on the Planck function through the temperature, and we can rewrite Eq. (477) as

$$\vec{F}(\vec{r},\nu,t) = -\frac{4\pi}{3\sigma(\vec{r},\nu,t)} \frac{\partial B(\nu,T)}{\partial T} \vec{\nabla}T(\vec{r},t) \quad . \tag{478}$$

Use of Eqs. (476) and (478) in Eq. (474) gives

$$I(\vec{r},\nu,\vec{\Omega},t) = B(\nu,T) - \frac{1}{\sigma(\vec{r},\nu,t)} \frac{\partial B(\nu,T)}{\partial T} \vec{\Omega}\cdot\vec{\nabla}T(\vec{r},t) \quad . \tag{479}$$

Equation (479) is the primary result of the equilibrium diffusion approximation and allows an explicit (albeit approximate) calculation of the specific intensity once the temperature distribution is known.

The quantities of primary interest in radiation hydrodynamic problems are the radiative energy density, flux, and pressure tensor as defined by Eqs. (21) through (23). In equilibrium diffusion theory these are given by, from Eq. (479),

$$E(\vec{r},t) = \frac{4\pi}{c} \int_0^\infty d\nu B(\nu,T) = aT^4(\vec{r},t) \quad , \tag{480}$$

$$\vec{F}(\vec{r},t) = -\frac{4\pi}{3} \vec{\nabla} T(\vec{r},t) \int_0^\infty d\nu \frac{1}{\sigma(\vec{r},\nu,t)} \frac{\partial B(\nu,T)}{\partial T} \quad , \tag{481}$$

$$p \equiv p_{ii} \doteq \frac{4\pi}{3c} \int_0^\infty d\nu B(\nu,T) = \frac{1}{3} aT^4(\vec{r},t) \quad , \tag{482a}$$

$$p_{ij} = 0 \quad , \quad i \neq j \quad , \tag{482b}$$

where a is the radiation constant given by Eq. (27).

The expression for the radiative flux, Eq. (481), is frequently written in a somewhat different form. By grouping terms, this equation can be rewritten as

$$\vec{F}(\vec{r},t) = -\frac{4\pi}{3} \vec{\nabla} T(\vec{r},t) \left[\frac{\int_0^\infty d\nu \frac{1}{\sigma(\nu)} \frac{\partial B(\nu,T)}{\partial T}}{\int_0^\infty d\nu \frac{\partial B(\nu,T)}{\partial T}} \right] \int_0^\infty d\nu \frac{\partial B(\nu,T)}{\partial T} \, . \tag{483}$$

If we recognize that

$$\int_0^\infty d\nu \frac{\partial B(\nu,T)}{\partial T} = \frac{ac}{\pi} T^3 \quad , \tag{484}$$

and define a mean or average (over frequency) total cross section $\sigma_R(\vec{r},t)$ as

$$\sigma_R = \frac{\int_0^\infty d\nu \, \frac{\partial B(\nu,T)}{\partial T}}{\int_0^\infty d\nu \, \frac{1}{\sigma(\nu)} \, \frac{\partial B(\nu,T)}{\partial T}} \quad , \tag{485}$$

Eq. (483) can be rewritten as

$$\vec{F}(\vec{r},t) = -\frac{4ac}{3} \frac{1}{\sigma_R(\vec{r},t)} T^3(\vec{r},t)\vec{\nabla}T(\vec{r},t) \quad , \tag{486}$$

or

$$\vec{F}(\vec{r},t) = -\frac{ac}{3\sigma_R(\vec{r},t)} \vec{\nabla}T^4(\vec{r},t) = -\frac{c}{3\sigma_R(\vec{r},t)} \vec{\nabla}E(\vec{r},t) \quad . \tag{487}$$

The coefficient $\sigma_R(\vec{r},t)$ is generally referred to as the Rosseland mean, and is widely used in radiative transfer work as we shall discuss in some detail later.

The (non-relativistic) hydrodynamic equations, with radiation terms, in the equilibrium diffusion approximation, result (presumably) from using these results for E, $\vec{F}$, and $\vec{P}$ in the equations of hydrodynamics given by Eqs. (16) through (18). This gives

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \quad , \tag{488}$$

$$\frac{\partial}{\partial t} (\rho\vec{u}) + \vec{\nabla}(P_m + P) + \vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = 0 \quad , \tag{489}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 + E_m + E\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2} \rho u^2 + E_m + P_m\right)\vec{u}\right]$$

$$= \vec{\nabla}\cdot\frac{c}{3\sigma_R} \vec{\nabla}E \quad , \tag{490}$$

where

$$E = 3P = aT^4 \quad . \tag{491}$$

In writing Eq. (489) we have neglected a term $\partial[\vec{F}/c^2]/\partial t$ as being negligibly small compared to $\partial(\rho\vec{u})/\partial t$.

This set of Eulerian equations can be put in another form by introducing the Lagrangian derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} \quad , \tag{492}$$

and deleting the time derivative of the kinetic energy in Eq. (490). This is accomplished by dotting Eq. (489) with $\vec{u}$ to form the mechanical energy balance and subtracting this result from Eq. (490). The result of these algebraic manipulations is the equivalent set of equations

$$\rho \frac{D}{Dt} \left(\frac{1}{\rho}\right) - \vec{\nabla}\cdot\vec{u} = 0 \quad , \tag{493}$$

$$\rho \frac{D\vec{u}}{Dt} + \vec{\nabla}(P_m + P) = 0 \quad , \tag{494}$$

$$\rho \frac{D}{Dt} \left[\frac{1}{\rho} (E_m + E)\right] + \rho(P_m + P) \frac{D}{Dt} \left(\frac{1}{\rho}\right)$$

$$= \vec{\nabla}\cdot\frac{c}{3\sigma_R} \vec{\nabla}E + \vec{\nabla}\cdot[(E + P)\vec{u}] \quad . \tag{495}$$

Equations (493) through (495) are <u>presumably</u> the equations of radiation-hydrodynamics in the equilibrium diffusion limit. However, <u>they are not</u>! This set of equations is <u>incorrect</u>. The correct equations are:

$$\rho \frac{D}{Dt} \left(\frac{1}{\rho}\right) - \vec{\nabla}\cdot\vec{u} = 0 \quad , \tag{496}$$

$$\rho \frac{D\vec{u}}{Dt} + \vec{\nabla}(P_m + P) = 0 \quad , \tag{497}$$

$$\rho \frac{D}{Dt}\left[\frac{1}{\rho}(E_m + E)\right] + \rho(P_m + P)\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \vec{\nabla}\cdot\frac{c}{3\sigma_R}\vec{\nabla}E \quad . \qquad (498)$$

Comparing Eqs. (496) through (498), the correct set, with Eqs. (493) through (495), the incorrect set, we see that it is the energy equation, Eq. (495), which is incorrect. It contains a term $\vec{\nabla}\cdot[(E + P)\vec{u}]$ not present in the correct equation.

That Eqs. (496) through (498) constitute, in fact, the correct set can be argued on physical grounds as follows. Equilibrium diffusion theory corresponds to the assumption of local thermodynamic equilibrium between the radiation and the matter. Thus in the hydrodynamic equations formulated as force and energy balances in the reference frame moving with the fluid (i.e., formulated with Lagrangian time derivatives), one would account for the radiation energy and pressure terms by simply adding these terms to the corresponding material terms. This is seen to be the case in Eqs. (497) and (498). The only other effect of radiation is the flow of energy by radiative processes, and this is accounted for by the diffusion term on the right hand side of Eq. (498).

Since Eqs. (496) through (498) are the correct equations, we should be able to obtain them in a consistent mathematical treatment. This, of course, implies that the derivation which led to Eqs. (493) through (495) is wrong. The error in this derivation was the use of an incorrect equation of transfer. We used the non-relativistic equation of transfer, Eq. (303), as our starting point. To obtain the correct equation, Eq. (498), it is necessary to use the relativistic equation of transfer containing certain $\vec{u}/c$ terms.

To demonstrate this explicitly, we rederive equilibrium diffusion theory starting with the relativistic equation of transfer. For simplicity of exposition, we neglect scattering, in which case the equation of transfer is simply

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \sigma(\nu,\vec{\Omega})[B(\nu,\vec{\Omega}) - I(\nu,\vec{\Omega})] \quad , \qquad (499)$$

where, according to Eqs. (253) and (254) [or (255)], we have, to first order in $\vec{u}/c$,

$$B(\nu,\vec{\Omega}) = B_o(\nu) + \vec{\Omega}\cdot\frac{\vec{u}}{c}\left[3B_o(\nu) - \nu\frac{\partial B_o(\nu)}{\partial\nu}\right] \quad , \tag{500}$$

$$\sigma(\nu,\vec{\Omega}) = \sigma_o(\nu) - \vec{\Omega}\cdot\frac{\vec{u}}{c}\left[\sigma_o(\nu) + \nu\frac{\partial\sigma_o(\nu)}{\partial\nu}\right] \quad . \tag{501}$$

Thus Eq. (499) becomes, correct to first order in $\vec{u}/c$,

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \left\{\sigma_o(\nu) - \vec{\Omega}\cdot\frac{\vec{u}}{c}\left[\sigma_o(\nu) - \nu\frac{\partial\sigma_o(\nu)}{\partial\nu}\right]\right\}\cdot$$

$$\cdot\left\{B_o(\nu) + \vec{\Omega}\cdot\frac{\vec{u}}{c}\left[3B_o(\nu) - \nu\frac{\partial B_o(\nu)}{\partial\nu}\right] - I(\nu,\vec{\Omega})\right\} \quad . \tag{502}$$

Forming the first two angular moments of Eq. (502), we obtain

$$\frac{\partial E(\nu)}{\partial t} + \vec{\nabla}\cdot\vec{F}(\nu) = \sigma_o(\nu)\left[4\pi B_o(\nu) - cE(\nu)\right]$$

$$+ \left[\sigma_o(\nu) + \nu\frac{\partial\sigma_o(\nu)}{\partial\nu}\right]\frac{\vec{u}}{c}\cdot\vec{F}(\nu) \quad , \tag{503}$$

$$\frac{1}{c}\frac{\partial\vec{F}(\nu)}{\partial t} + \vec{\nabla}\cdot c\vec{P}(\nu) + \sigma_o(\nu)\vec{F}(\nu)$$

$$= \frac{4\pi}{3}\sigma_o(\nu)\left[3B_o(\nu) - \nu\frac{\partial B_o(\nu)}{\partial\nu}\right]\frac{\vec{u}}{c}$$

$$- \left[\frac{4\pi}{3}B_o(\nu) - \frac{\vec{u}}{c}\cdot c\vec{P}(\nu)\right]\left[\sigma_o(\nu) + \nu\frac{d\sigma_o(\nu)}{d\nu}\right] \quad . \tag{504}$$

In Eq. (503) we recognize that we can always neglect the $(\vec{u}/c)\cdot\vec{F}(\nu)$ term since $|\vec{F}(\nu)| \leq cE(\nu)$, and we are assuming $\vec{u}/c$ small. If we introduce the Eddington approximation, namely

$$\vec{\nabla}\cdot\vec{P}(\nu) = \frac{1}{3}\,\vec{\nabla}E(\nu) \; ; \quad \vec{u}\cdot\vec{P}(\nu) = \frac{1}{3}\,\vec{u}E(\nu) \quad , \tag{505}$$

and neglect the $\partial\vec{F}/\partial t$ term (by assuming the problem is collision dominated), we then have the two moment equations

$$\frac{\partial E(\nu)}{\partial t} + \vec{\nabla}\cdot\vec{F}(\nu) = \sigma_o(\nu)\left[4\pi B_o(\nu) - cE(\nu)\right] \quad , \tag{506}$$

$$\frac{1}{3}\,\vec{\nabla}(cE) + \sigma_o(\nu)\vec{F}(\nu) = \frac{4\pi}{3}\,\sigma_o(\nu)\left[3B_o(\nu) - \nu\,\frac{\partial B_o(\nu)}{\partial \nu}\right]\frac{\vec{u}}{c}$$

$$- \frac{1}{3}\left[\sigma_o(\nu) + \nu\,\frac{\partial \sigma_o(\nu)}{\partial \nu}\right]\left[4\pi B_o(\nu) - cE(\nu)\right]\frac{\vec{u}}{c} \quad . \tag{507}$$

To complete the equilibrium diffusion approximation it is assumed that locally radiation emission and absorption at each frequency are in equilibrium. This implies that the right hand side of Eq. (506) should be set to zero, i.e.,

$$cE(\nu) = 4\pi B_o(\nu) \quad . \tag{508}$$

Using this result in Eq. (507), we obtain

$$\vec{F}(\nu) = -\,\frac{4\pi}{3\sigma_o(\nu)}\,\frac{\partial B_o(\nu)}{\partial T}\,\vec{\nabla}T + \frac{4\pi}{3}\left[3B_o(\nu) - \nu\,\frac{\partial B_o(\nu)}{\partial \nu}\right]\frac{\vec{u}}{c} \quad . \tag{509}$$

Integration of Eq. (509) over all frequencies, introducing

$$E = 3P = aT^4 \quad , \tag{510}$$

yields

$$\vec{F} = -\frac{c}{3\sigma_R}\vec{\nabla}E + \frac{4}{3}E\vec{u} \quad , \tag{511}$$

or, equivilently,

$$\vec{F} = -\frac{c}{3\sigma_R}\vec{\nabla}E + (E + P)\vec{u} \quad . \tag{512}$$

The convective term in Eq. (512), i.e., that involving the velocity $\vec{u}$ is the result of having retained the relativistic terms in the equation of transfer. This term is clearly seen to be formally of the order $\vec{u}/c$ compared to the gradient term. Because of this, the terms in the equation of transfer which give rise to the velocity term in Eq. (512) [that is, the $\vec{u}/c$ terms in Eq. (502)] can generally be dropped as negligibly small if, in fact, $\vec{u}/c$ is negligibly small. However, in a true equilibrium diffusion problem, the two terms on the right hand side of Eq. (512) can be of comparable magnitude, since it is the essence of an equilibrium diffusion problem that the gradient term in Eq. (512) is small. Further, this additional term is needed to obtain the correct energy equation, Eq. (498).

If we use Eq. (512) for $\vec{F}$ in the general non-relativistic energy equation, Eq. (18), we find

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + E_m + E\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + E_m + P_m + E + P\right)\vec{u}\right]$$

$$= \vec{\nabla}\cdot\frac{c}{3\sigma_R}\vec{\nabla}E \quad . \tag{513}$$

Combining this with the momentum equation, Eq. (489) to eliminate $\partial(\rho u^2/2)\partial t$ and introducing the Lagrangian time derivative, Eq. (513) can be rewritten as

$$\rho \frac{D}{Dt}\left[\frac{1}{\rho}(E_m + E)\right] + \rho(P_m + P)\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \vec{\nabla}\cdot\frac{c}{3\sigma_R}\vec{\nabla}E \quad , \tag{514}$$

which is the correct energy equation [compare Eqs. (498) and (514)].

In summary, in any problem other than one described by the equilibrium diffusion limit, one can, assuming $\vec{u}/c$ small, properly neglect $\vec{u}/c$ terms in the equation of transfer. However, if one wants the general equation of transfer to properly limit to the equilibrium diffusion description, one needs carry $\vec{u}/c$ terms. [It should be noted that in the material terms in the relativistic hydrodynamic equations, the lowest order corrections are $O(u^2/c^2)$]. It should also be noted that the $\vec{u}/c$ correction terms to the material rest frame absorption cross section cancelled out in the development of the equilibrium diffusion limit. This suggests a simpler equation of transfer, namely

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega})$$

$$= \sigma_o(\nu)\left\{B_o(\nu) + \vec{\Omega}\cdot\frac{\vec{u}}{c}\left[3B_o(\nu) - \nu\frac{\partial B_o(\nu)}{\partial\nu}\right] - I(\nu,\vec{\Omega})\right\} \quad . \tag{515}$$

In the non-diffusion limit, this is a correct equation since the $\vec{u}/c$ terms can properly be neglected and hence the exact form of these terms is irrelevant. In the diffusion limit, this equation of transfer is also correct in that it again leads to Eq. (512). An even simpler equation with these same properties is

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega})$$

$$= \sigma_o(\nu)\left\{B_o(\nu)\left[1 + 4\vec{\Omega}\cdot\frac{\vec{u}}{c}\right] - I(\nu,\vec{\Omega})\right\} \quad . \tag{516}$$

Obviously, other forms are also possible. Thus, with respect to the derivation of the correct equilibrium diffusion description of radiation hydrodynamics, Eqs. (502), (515), and (516), as well as other possibilities, should be equally valid. Equation (502) is clearly the most complex, although asthetically it is to be preferred since it follows most directly from basic physical considerations.

E. Marshak Waves

An interesting physical phenomenon in radiative transfer is that of Marshak waves. If a local source of energy is introduced into a cold absorber, and the only mechanism for energy transfer is via radiative processes, the bulk of the energy propagates as a thermal wave. Ahead of the wavefront (distinct from the speed of light wavefront), the material temperature is essentially zero. This phenomenon is described remarkably accurately by equilibrium diffusion theory.

We consider uniform matter with a constant heat capacity c_V so that the material energy density is given by

$$E_m = c_V T \quad . \tag{517}$$

We assume that the scattering cross section is zero, and that the absorption cross section is proportional to ν^{-s} (for real cross sections, an idealization is $s \approx 3$). Then the Rosseland mean, σ_R, defined by Eq. (485), will be proportional to T^{-s}, and we write

$$\sigma_R = \frac{b}{T^s} \quad , \tag{518}$$

where b is a constant. Neglecting hydrodynamic motion, and assuming the material energy density dominates the radiative energy density, the equilibrium diffusion theory energy equation, Eq. (513) or (514), becomes

$$c_v \frac{\partial T}{\partial t} = \vec{\nabla} \cdot \frac{cT^s}{3b} \vec{\nabla} aT^4 \quad , \tag{519}$$

or

$$\frac{\partial T}{\partial t} = K \vec{\nabla} \cdot T^n \vec{\nabla} T \quad , \tag{520}$$

where K is a composite constant

$$K = \frac{4ac}{3bc_v} \quad , \tag{521}$$

and $n = s + 3$. We consider two problems described by Eq. (520) and demonstrate the existence, according to this model, of Marshak waves.

Problem #1

This problem corresponds to an instantaneous release, at $t = 0$, of an amount of energy $c_v Q$ [we include the factor c_v here to simplify a subsequent formula, namely Eq. (525)] at a point $r = 0$ in an ℓ^{th} dimensional infinite medium. Because of the symmetry of this problem, Eq. (520) becomes one dimensional, i.e.,

$$\frac{\partial T}{\partial t} = \frac{K}{r^{\ell-1}} \frac{\partial}{\partial r} \left(r^{\ell-1} T^n \frac{\partial T}{\partial r} \right) \, , \quad t > 0 \quad , \tag{522}$$

where $\ell = 1,2,3$ corresponds to plane, cylindrical, and spherical geometry, respectively. The initial and boundary conditions on Eq. (522) are

$$T(r,0) = 0 \quad , \tag{523}$$

$$T(\infty,t) = 0 \; ; \quad T(0,t) < \infty \quad . \tag{524}$$

[In plane geometry, $\ell = 1$, Eq. (524) is replaced with $T(-\infty,t) = T(\infty,t) = 0$]. The source in this problem implies a Dirac delta function, in both space and time, source should be added to the right hand side of Eq. (522). An equivalent treatment is to leave Eq. (522) as it is, and impose the integral energy conservation condition

$$Q = \int_0^\infty dr A_\ell(r) T(r,t) \quad , \tag{525}$$

where $A_\ell(r)$ is the surface area of a sphere of radius r in ℓth dimensional space, i.e.,

$$A_\ell(r) = \frac{2\pi^{\ell/2}}{\Gamma(\ell/2)} r^{\ell-1} \quad , \tag{526}$$

with $\Gamma(z)$ denoting the usual gamma function.

We seek a similarity solution of the form

$$T(r,t) = \left[\frac{Q^2}{(Kt)^\ell}\right]^{\left(\frac{1}{2+n\ell}\right)} f(\xi) \quad , \tag{527}$$

where the similarity variable ξ is defined as

$$\xi = r\left[Q^n Kt\right]^{-\left(\frac{1}{2+n\ell}\right)} \quad . \tag{528}$$

Then Eq. (522) becomes

$$\left(\frac{2+n\ell}{n+1}\right) \frac{d}{d\xi} \left(\xi^{\ell-1} \frac{df^{n+1}}{d\xi}\right) + \frac{d}{d\xi} \left(\xi^\ell f\right) = 0 \quad , \tag{529}$$

and the subsidiary conditions, Eq. (523) through (525), become

$$f(0) < \infty ; \quad f(\infty) = 0 \quad , \tag{530}$$

$$1 = \frac{2\pi^{\ell/2}}{\Gamma(\ell/2)} \int_0^\infty d\xi \ \xi^{\ell-1} f(\xi) \quad . \tag{531}$$

[We note that $f(\infty) = 0$ is redundant since it is implied in Eq. (528)].

A first integration of Eq. (529) gives

$$\left(\frac{2 + n\ell}{n + 1}\right) \xi^{\ell-1} \frac{df^{n+1}}{d\xi} + \xi^{\ell} f = 0 \quad , \tag{532}$$

where the constant of integration has been set to zero by considerations at $\xi = 0$ (i.e., at $r = 0$). A second integration gives

$$f(\xi) = \left[\frac{n}{2(2 + n\ell)} (\xi_o^2 - \xi^2)\right]^{1/n} \quad , \tag{533}$$

where ξ_o^2 is a constant of integration. For the solution to be well behaved at ∞ (namely, vanish), we take Eq. (533) to be the solution for $f(\xi)$ for $0 \leq \xi \leq \xi_o$, and set

$$f(\xi) \equiv 0 \ , \quad \xi \geq \xi_o \quad . \tag{534}$$

It is easily verified that this solution has the proper continuity conditions at $\xi = \xi_o$ as required by Eq. (529), namely $f(\xi)$ and $df^{n+1}/d\xi$ are continuous at ξ_o. Thus Eq. (533) must yield

$$f(\xi_o) = \left.\frac{df^{n+1}}{d\xi}\right|_{\xi=\xi_o} = 0 \quad , \tag{535}$$

and it does. We determine the constant ξ_o from the integral conservation equation, Eq. (531). This gives

$$\xi_o^{(2+n\ell)} = \frac{(2+n\ell)^{(1+n)}\, 2^{(1-n)}\, \Gamma^n\left(\frac{\ell}{2}+\frac{1}{n}\right)}{n\pi^{n\ell/2}\, \Gamma^n\left(\frac{1}{n}\right)} \quad . \tag{536}$$

Combining all of these results, we obtain as the solution for the temperature

$$T(r,t) = 0 \ , \quad r > r_o(t) \ , \tag{537a}$$

$$T(r,t) = \left[\frac{n}{2(2+n\ell)} \frac{r_o^2(t) - r^2}{Kt}\right]^{1/n} \ , \quad r \leq r_o(t) \ , \tag{537b}$$

where the position of the wavefront, $r_o(t)$, is given by

$$r_o(t) = \left[\frac{(2+n\ell)^{(1+n)}\, 2^{(1-n)}\, \Gamma^n\left(\frac{\ell}{2}+\frac{1}{n}\right) Q^n Kt}{n\pi^{n\ell/2}\, \Gamma^n\left(\frac{1}{n}\right)}\right]^{\left(\frac{1}{2+n\ell}\right)} . \tag{538}$$

We note that for n = 6, a reasonable value for realistic cross sections, that T(r,t) is essentially flat behind the wavefront. We also note that for this value of n, the wavefront moves quite slowly, i.e.,

$$r_o(t) \propto \begin{cases} t^{1/8} & \text{(planes)} \\ t^{1/14} & \text{(cylinders)} \\ t^{1/20} & \text{(spheres)} \end{cases} \quad . \tag{539}$$

For n = 0, the original equation, Eq. (522), is linear. In this case, as is well known, there is no wavefront, i.e.,

$$r_o(t) \to \infty \ , \tag{540}$$

and the solution, Eq. (537), limits to

$$T(r,t) = \frac{Q}{2^{\ell}(\pi Kt)^{\ell/2}} \exp - (r^2/4Kt) \quad , \tag{541}$$

the well known heat transfer conduction result.

Problem #2

This problem corresponds to a source free halfspace occupying $0 \leq z < \infty$, with a prescribed temperature T_0 applied at the boundary at $z = 0$ for all $t > 0$. The equilibrium diffusion equation, Eq. (522), becomes, since $\ell = 1$,

$$\frac{\partial T}{\partial t} = K \frac{\partial}{\partial z} T^n \frac{\partial T}{\partial z} \quad , \tag{542}$$

with initial and boundary conditions

$$T(z,0) = 0 \quad , \tag{543}$$

$$T(0,t) = T_o \ ; \quad T(\infty,t) = 0 \quad . \tag{544}$$

In this case an appropriate similarity solution is

$$T(z,t) = g(\eta) \quad , \tag{545}$$

with

$$\eta = \frac{z}{\sqrt{t}} \quad . \tag{546}$$

Then Eqs. (542) through (544) become

$$\frac{\eta}{2} \frac{\partial g}{\partial \eta} = K \frac{\partial}{\partial \eta} g^n \frac{\partial g}{\partial \eta} \quad , \tag{547}$$

$$g(0) = T_o \ ; \quad g(\infty) = 0 \quad . \tag{548}$$

Unfortunately, Eq. (547) and (548) do not admit a simple closed form solution. However, by examining this equation at $\eta = \infty$, one can demonstrate that $g(\eta) \equiv 0$ for sufficiently large η. Hence a wavefront exists, and we again obtain a Marshak thermal wave. We note that in this case, Eq. (546) implies that the position of the wavefront is proportional to $\sqrt{t}$, a much faster wave propagation than in the previous problem.

High Order Approximations

The diffusion approximations to the equation of transfer we have discussed have one overriding characteristic in common: they are all of limited accuracy. If, for a given problem, their error is unacceptable, there is no way, within the framework of the approximations, to systematically improve their accuracy.

We now briefly discuss three types of approximations to the equation of transfer which are capable of estimating the solution to the equation of transfer to within any desired accuracy criteria. These are:

1. The Spherical Harmonic (P-N) Method;
2. The Discrete Ordinate (S-N) Method;
3. The Monte Carlo Method.

In radiation-hydrodynamic problems, the Monte Carlo method has been used much more than the P-N or S-N method.

F. The Spherical Harmonic (P-N) Method

The basis of this method is the expansion of the specific intensity in a complete set of angular functions, called spherical harmonics, or surface harmonics. These are

$$Y_n^m(\hat{\Omega}) = Y_n^m(\mu,\phi) = P_n^{|m|}(\mu)e^{im\phi} \quad , \tag{549}$$

where the $p_n^m(\mu)$ are the associated Legendre functions defined as

$$P_n^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m Pn(\mu)}{d\mu^m} , \quad 0 \leq m \leq n , \tag{550}$$

and $P_n(\mu)$ is the usual Legendre polynomial. These functions are complete on the unit sphere, which means that any function defined in the intervals $-1 \leq \mu \leq 1$ and $0 \leq \phi \leq 2\pi$ can be expanded as

$$f(\vec{\Omega}) = f(\mu,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m f_n^m Y_n^m(\vec{\Omega}) , \tag{551}$$

where the f_n^m are the expansion coefficients and the a_n^m are the normalization coefficients associated with the orthoganality condition

$$\int_{4\pi} d\vec{\Omega} \, Y_n^m(\vec{\Omega}) \, Y_j^{k*}(\vec{\Omega}) = \frac{\delta_{nj}\delta_{mk}}{a_n^m} , \tag{552}$$

where the asterisk implies complex conjucate. Explicitly, we have

$$a_n^m = \frac{1}{4\pi} \left[\frac{(2n + 1)(n - |m|)!}{(n + |m|)!}\right] . \tag{553}$$

Using the orthogonality relationship, we have

$$f_n^m = \int_{4\pi} d\vec{\Omega} \, Y_n^{m*}(\vec{\Omega}) f(\vec{\Omega}) . \tag{554}$$

The basis of the P-N approximation is to expand the specific intensity of radiation in spherical harmonic according to

$$I(\vec{r},\nu,\vec{\Omega},t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m I_n^m (\vec{r},\nu,t) Y_n^m(\vec{\Omega}) \quad , \tag{555}$$

where

$$I_n^m (\vec{r},\nu,t) = \int_{4\pi} d\vec{\Omega}\; Y_n^{m*}(\vec{\Omega}) I(\vec{r},\nu,\vec{\Omega},t) \quad . \tag{556}$$

This leads to an infinite set of coupled equations for the expansion coefficients I_n^m. The P-N method consists of truncating this set of equations by setting

$$I_n^m = 0 \ , \quad n > N \quad . \tag{557}$$

We will see that the P-1 approximation is, in fact, the Eddington description, in telegrapher's form, we have already discussed. We also note that since the functions $Y_n^m(\vec{\Omega})$ are complete, the P-N method approaches exactness as $N \rightarrow \infty$. Hence the P-N method can be considered as the systematic extension of the Eddington approximation to the higher order descriptions.

We note also that the radiative energy density and three components of the radiative flux are related to the first four expansion coefficients according to

$$E = \frac{1}{c} I_o^o \quad , \tag{558}$$

$$F_x = \frac{1}{2} [I_1^{-1} + I_1^1] \quad , \tag{559}$$

$$F_y = \frac{1}{2i} [I_1^{-1} - I_1^1] \quad , \tag{560}$$

$$F_z = I_1^0 \quad . \tag{561}$$

Similarly, the components of the pressure are linear combinations of I_o^o and I_2^m.

We carry out the details of the P-N method only in plane geometry. Other geometries are treated in a similar fashion, but the algebraic details are somewhat more complex. In plane geometry, all of the I_n^m are zero if $m \neq 0$ since we have azimuthal symmetry. In this case, Eq. (555) becomes

$$I(z,\nu,\mu,t) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{4\pi}\right) I_n(z,\nu,t) P_n(\mu) \quad , \tag{562}$$

where we have set $I_n^o = I_n$. In plane geometry, the transport equation we are considering, Eq. (303), is

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu \frac{\partial I(\mu)}{\partial z} + \sigma I(\mu) = \sigma_a B + \frac{\sigma_s}{4\pi} I_o \quad , \tag{563}$$

where we have set $cE = I_o$. We use Eq. (562) in Eq. (563), employ the recurrence relationship

$$\mu P_n(\mu) = \left(\frac{n+1}{2n+1}\right) P_{n+1}(\mu) + \left(\frac{n}{2n+1}\right) P_{n-1}(\mu) \quad , \tag{564}$$

and equate the coefficients of $P_n(\mu)$ to obtain the infinite set of equations for the expansion coefficients

$$\frac{1}{c}\frac{\partial I_o}{\partial t} + \frac{\partial I_1}{\partial z} + \sigma_a I_o = 4\pi\sigma_a B \quad , \tag{565}$$

$$\frac{2n+1}{c}\frac{\partial I_n}{\partial t} + n\frac{\partial I_{n-1}}{\partial z} + (2n+1)\sigma I_n + (n+1)\frac{\partial I_{n+1}}{\partial z} = 0$$

$$n \geq 1 \quad , \tag{566}$$

where $\sigma_a \equiv \sigma - \sigma_s$.

Let us consider Eq. (565) and the first N equations of the infinite set given by Eq. (566). This amounts to N + 1 equations in N + 2 unknowns, namely $I_0, I_1, \ldots, I_{N+1}$. Hence it is necessary in order to close this set to somehow reduce the number of unknowns by one. Since Eq. (562) is presumed to be a convergent expansion, the I_n must decrease with increasing n, and the natural and simplest truncation procedure is to make the approximation

$$I_{N+1}(z,\nu,t) = 0 \quad . \tag{567}$$

We then have as the last equation in the P-N set

$$\frac{2N+1}{c}\frac{\partial I_N}{\partial t} + N\frac{\partial I_{N-1}}{\partial z} + (2N+1)\sigma I_N = 0 \quad . \tag{568}$$

Equation (565), the first N-1 equation of Eq. (566), and Eq. (568) constitute N + 1 equations in N + 1 unknowns $I_0, I_1, \ldots, I_N$, and are the equations of the N^{th} order P-N approximation in plane geometry.

As N becomes infinite, the solution of the P-N equations approaches the solution of the equation of transfer. However, experience shows that even in very low order the P-N method is quite accurate. For example, for N = 1 we have as the P-N equations

$$\frac{1}{c}\frac{\partial I_o}{\partial t} + \frac{\partial I_1}{\partial z} + \sigma_a I_o = 4\pi\sigma_a B \quad , \tag{569}$$

$$\frac{3}{c}\frac{\partial I_1}{\partial t} + \frac{\partial I_o}{\partial z} + 3\sigma I_1 = 0 \quad . \tag{570}$$

Recalling that $I_o = cE$ and $I_1 = F$, these are just the Eddington equations, Eqs. (306) and (309). It is well known that the even order P-N approximations, (i.e., N = 2,4,...) have difficulties which are not suffered in odd order, and that the succeeding even order approximation is generally less accurate than the one lower odd order approximation. For these reasons, only odd order approximations are used in practice, and for the remainder of our discussion we restrict our attention to N odd. We note that Eq. (565) is extant in all P-N approximations, i.e., for any N. Since this is just the continuity (conservation) equation, the P-N method is conservative of photons.

The initial conditions for the P-N approximation follow immediately from the initial condition on the equation of transfer, Eq. (86). We have

$$I_n(z,\nu,0) = 2\pi \int_{-1}^{1} d\mu P_n(\mu)\Lambda(z,\nu,\mu) \ , \quad n = 1,2,\ldots,N \ . \qquad (571)$$

Concerning boundary conditions, we require (N + 1)/2 conditions on each face of the planar system. If we consider the left hand surface, say $z = z_\ell$, these conditions can be taken as (N + 1)/2 weighted averages of the exact boundary condition, Eq. (85) That is, we write

$$2\pi \int_{0}^{1} d\mu\ W_m(\mu)\left[I(z_\ell,\nu,\mu,t) - \Gamma(z_\ell,\nu,\mu,t)\right] = 0 \ ,$$

$$m = 1,2,\ldots,(N + 1)/2 \ , \qquad (572)$$

where the $W_m(\mu)$, arbitrary linearly independent functions, are the weight functions. Since the P-N approximation consists of setting $I_n(z,\nu,t) = 0$ for $n > N$, we use a truncated version of Eq. (562) in Eq. (572) and write

$$\sum_{n=0}^{N} \left(\frac{2n+1}{2}\right) I_n(z_\ell,\nu,t) \int_{0}^{1} d\mu W_m(\mu) P_n(\mu)$$

$$= 2\pi \int_{0}^{1} d\mu W_m(\mu)\Gamma(z_\ell,\nu,\mu,t) \ , \quad m = 1,2,\ldots,(N + 1)/2 \ . \qquad (573)$$

Once the $W_m(\mu)$ have been specified, Eq. (573) is the required $(N + 1)/2$ relationships among the $I_n(z,\nu,t)$ at $z = z_\ell$. The so-called Marshak or Milne boundary conditions consists of the choice

$$W_m(\mu) = P_{2m-1}(\mu) \ , \quad m = 1,2,\ldots(N + 1)/2 \ , \tag{574}$$

or equivalently

$$W_m(\mu) = \mu^{2m-1} \ , \quad m = 1,2,\ldots(N + 1)/2 \ . \tag{575}$$

The Marshak/Milne condition for $m = 1$ has the physical interpretation of preserving the incoming flux per unit frequency. The Mark boundary conditions correspond to choosing the weight functions as Dirac delta functions

$$W_m(\mu) = \delta(\mu - \mu_m) \ , \quad m = 1,2,\ldots(N + 1)/2 \ , \tag{576}$$

where the μ_m are the positive roots of the $(N + 1)^{th}$ Legendre polynomial, i.e.,

$$P_{N+1}(\mu_m) = 0 \ , \quad \mu_m > 0 \ , \quad m = 1,2,\ldots(N + 1)/2 \ . \tag{577}$$

It can be shown that in the special case of no incoming flux $(\Gamma = 0)$, the Mark conditions are equivalent to surrounding the system with a source free, pure absorber and carrying out the P-N calculation over all space, assuming $I_n(z,\nu,t)$ to vanish at $z = -\infty$. Similar considerations give the Marshak/Milne and Mark boundary conditions at the right hand surface of a planar system. The only difference is that the integrals in Eq. (573) cover the range $-1 \leq \mu < 0$ rather than $0 \leq \mu \leq 1$, and one uses the negative, rather than the positive roots from Eq. (577). In practice, the Marshak/Milne conditions generally prove to be more accurate than the Mark conditions.

It should be remarked that other truncation schemes, rather than setting $I_{N+1}(z,\nu,t) = 0$, have been proposed within the context of the P-N method. What is required in general to truncate the infinite set of equations is a method of relating $I_{N+1}(z,\nu,t)$ to the lower expansion coefficients. This can be done with some generality in the following way. Suppose that one expects the angular distribution to be approximately described by a specified function $f(z,\nu,\mu,t)$. For the purposes of truncation, we then assume that the specific intensity of radiation can be represented by

$$I(z,\nu,\mu,t) = \sum_{n=0}^{N-2} c_n(z,\nu,t)P_n(\mu)$$

$$+ c_d(z,\nu,t)f_o(z,\nu,\mu,t) + c_e(z,\nu,t)f_e(z,\nu,\mu,t) \quad , \qquad (578)$$

where f_o and f_e are the odd and even (in μ) parts of $f(z,\nu,\mu,t)$ respectively, and the c_n, c_d, and c_e are expansion coefficients. Multiplying Eq. (578) by $P_{N-1}(\mu)$ and integrating over all solid angle, we find (we assume N odd)

$$I_{N-1}(z,\nu,t) = 2\pi c_e(z,\nu,t) \int_{-1}^{1} d\mu P_{N-1}(\mu)f_e(z,\nu,\mu,t) \quad . \qquad (579)$$

Similarly, one can obtain an expression for $I_{N+1}(z,\nu,t)$. Taking the ratio of these two results, we obtain

$$I_{N+1}(z,\nu,t) = \left[\frac{\int_{-1}^{1} d\mu P_{N+1}(\mu)f_e(\mu)}{\int_{-1}^{1} d\mu P_{N-1}(\mu)f_e(\mu)}\right] I_{N-1}(z,\nu,t) \quad , \qquad (580)$$

as the truncation condition. Hence Eq. (566) for n = N becomes

$$\frac{2N+1}{c}\frac{\partial I_N}{\partial t} + \frac{\partial}{\partial z}\{[N + (N+1)R_N]I_{N-1}\}$$

$$+ (2N+1)\sigma I_N = 0 \quad , \tag{581}$$

where $R_N = R_N(z,\nu,t)$ is the ratio of integrals appearing in Eq. (580). The modified P-N approximation then consists of Eq. (565), the first N-1 equations of Eq. (566), and Eq. (581) as the truncating equation.

As an example of this type of truncation, we could choose $f(z,\nu,\mu,t)$ as the asymptotic distribution [see Eq. (330)]

$$f_e(\mu) = \frac{1}{1 - K^2\mu^2} \quad , \tag{582}$$

where K satisfies

$$\frac{2K}{\omega} = \ln\left(\frac{1+K}{1-K}\right) \quad . \tag{583}$$

[ω in Eq. (583) could also be $\tilde{\omega}$ or $\hat{\omega}$.] From Eq. (580) we then find

$$R_N \equiv \frac{I_{N+1}}{I_{N-1}} = \frac{Q_{N+1}(1/K)}{Q_{N-1}(1/K)} \quad , \tag{584}$$

where

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} d\xi \, \frac{P_n(\xi)}{z - \xi} \quad , \tag{585}$$

is the n^{th} order Legendre function of the second kind. This "asymptotic" P-N approximation reproduces correct asymptotic moments in all orders N when the specific intensity is, in fact, in an asymptotic state. In addition, for $N > 1$, this approximation will, in all orders, accurately describe the almost isotropic transport problem.

One could also envision using other angular distributions, such as those due to Minerbo and Levermore as previously discussed, to truncate the spherical harmonic equations. This procedure, in essence, extends the notion of flux limiters and Eddington factors to higher order (than diffusion) approximations to the equation of transfer.

The energy density, radiative flux, and pressure tensor needed in the hydrodynamic equations are given in the P-N or modified P-N method as

$$E(z,t) = \frac{1}{c} \int_0^\infty d\nu I_0(z,\nu,t) \quad , \tag{586}$$

$$F_z(z,t) = \int_0^\infty d\nu I_1(z,\nu,t) \quad , \tag{587}$$

$$F_x(z,t) = F_y(z,t) = 0 \quad , \tag{588}$$

$$p_{zz}(z,t) = \frac{1}{c} \int_0^\infty d\nu [\frac{1}{3} I_0(z,\nu,t) + \frac{2}{3} I_2(z,\nu,t)] \quad , \tag{589}$$

$$p_{xx}(z,t) = p_{yy}(z,t)$$

$$= \frac{1}{c} \int_0^\infty d\nu [\frac{1}{3} I_0(z,\nu,t) - \frac{1}{3} I_2(z,\nu,t)] \quad , \tag{590}$$

$$p_{ij}(z,t) = 0 \ , \quad i \neq j \ . \tag{591}$$

We note that

$$p_{xx} + p_{yy} + p_{zz} = E \quad , \tag{592}$$

a property true in general as discussed earlier.

G. The Discrete Ordinate (S-N) Method

One of the appealing features of the P-N method is that it is in fact an entire set of approximations, which, by choosing N large enough, can be used to estimate the solution to the equation of transfer to within an arbitrarily small error. Another set of approximations with the same feature is the discrete ordinate, or S-N, method, with N again denoting the order of the approximation. For N infinite, the S-N solution is the exact solution to the equation of transfer, just as in the P-N method.

Restricting our discussion to plane geometry, Eq. (303), the equation of transfer, is

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu\frac{\partial I(\mu)}{\partial z} + \sigma I(\mu) = \sigma_a B + \frac{\sigma_s}{2}\int_{-1}^{1} d\mu' I(\mu') \quad , \qquad (593)$$

where we have explicitly written the integration over angle in Eq. (593). The basis of the S-N method as applied to this equation of transfer is extremely simple. One uses an integration quadrature scheme to approximate the integrals over the μ (angle) variable. If we consider an N point scheme, denoting the quadrature points by μ_j and the corresponding weights by W_j, we make the replacement

$$\int_{-1}^{1} d\mu' I(\mu') \rightarrow \sum_{j=1}^{N} W_j I(\mu_j) \quad . \qquad (594)$$

The equation of transfer, Eq. (593), then becomes

$$\frac{1}{c}\frac{\partial I(\mu)}{\partial t} + \mu\frac{\partial I(\mu)}{\partial z} + \sigma I(\mu) = \sigma_a B + \frac{\sigma_s}{2}\sum_{j=1}^{N} W_j I(\mu_j) \quad . \qquad (595)$$

To obtain the S-N equations, one merely evaluates this equation at the quadrature points μ_i. Hence, the N^{th} order discrete ordinate approximation in plane geometry consists of the N equations

$$\frac{1}{c}\frac{\partial I(\mu_i)}{\partial t} + \mu_i \frac{\partial I(\mu_i)}{\partial z} + \sigma I(\mu_i) = \sigma_a B + \frac{\sigma_s}{2}\sum_{j=1}^{N} W_j I(\mu_j) \ ,$$

$$i = 1,2,\ldots N \ , \qquad (596)$$

for the N unknowns $I(z,\nu,\mu_i,t)$.

The N initial conditions required for the S-N approximation follow immediately from the initial condition on the equation of transfer, Eq. (86), by evaluating the condition at the quadrature points μ_i. We have

$$I(z,\nu,\mu_i,0) = \Lambda(z,\nu,\mu_i) \ , \quad 1 \leq i \leq N \ . \qquad (597)$$

At the left hand face of the planar system, say $z = z_\ell$, we obtain the N/2 boundary conditions required (we assume N even with an equal number of positive and negative μ_i) by evaluating the transport boundary condition, Eq. (85), at the positive quadrature points. This gives

$$I(z_\ell,\nu,\mu_i,t) = \Gamma(z_\ell,\nu,\mu_i,t) \ , \quad \mu_i > 0 \ . \qquad (598)$$

Similarly, at the right hand face, say $z = z_r$, we have

$$I(z_r,\nu,\mu_i,t) = \Gamma(z_r,\nu,\mu_i,t), \quad \mu_i < 0 \ , \qquad (599)$$

as the N/2 boundary conditions.

The quadrature scheme used in the S-N method is arbitrary, although it generally employs quadrature points which occur in pairs, one being the negative of the other. This retains the symmetry of the exact equation of transfer and leads to an equal number of boundary conditions on each face of the system as just discussed. Since the quadrature scheme is arbitrary, it can be chosen to accurately integrate the expected angular dependence of the specific intensity. For example, if the angular dependence is highly peaked around $\mu = \pm 1$, one can use a quadrature which has its points concentrated near the endpoints of the μ range $-1 \leq \mu \leq 1$. Alternately, one could use a quadrature scheme which would integrate exactly the asymptotic distribution. Or, one could base the quadrature points on an angular distribution such as that derived by Minerbo or Levermore, as previously discussed. This could be thought of as introducing flux limiting and/or variable Eddington factors into S-N calculations. If one has no apriori knowledge of the angular distribution, the usual choice is the Gauss-Legendre quadrature scheme, in which the μ_i are chosen as the zeros of the N^{th} Legendre polynomial, i.e.,

$$P_N(\mu_i) = 0 \quad . \tag{600}$$

This quadrature scheme integrates a polynomial in μ more accurately than any other quadrature scheme. With this choice for the quadrature points and the associated weights it is well known that the S-N method is closely related to the P-N method of one lower order (in systems with plane geometry). Many aspects of the Gauss-Legendre S-N approximation have been examined in detail in plane geometry by Chandrasekhar in his classic text "Radiative Transfer".

The energy density, radiative flux, and pressure tensor according to the S-N method are obtained by again employing the same quadrature scheme used in deriving the S-N equations to perform the integrations over the μ variable. We have

$$E(z,t) = \frac{2\pi}{c} \int_0^\infty d\nu \sum_{i=1}^{N} W_i I(z,\nu,\mu_i,t) \quad , \tag{601}$$

$$F_z(z,t) = 2\pi \int_0^\infty d\nu \sum_{i=1}^{N} W_i \mu_i I(z,\nu,\mu_i,t) \quad , \tag{602}$$

$$p_{zz}(z,t) = \frac{2\pi}{c}\int_0^\infty d\nu \sum_{i=1}^{N} W_i \mu_i^2 I(z,\nu,\mu_i,t) \quad , \tag{603}$$

$$p_{xx}(z,t) = p_{yy}(z,t)$$

$$= \frac{\pi}{c}\int_0^\infty d\nu \sum_{i=1}^{N} W_i (1 - \mu_i^2) I(z,\nu,\mu_i,t) \quad . \tag{604}$$

The other components of the radiative flux and pressure tensor are zero.

The modern S-N method in other geometries, in particular in curvilinear systems, is much more involved than it is in plane geometry. Specifically, the angular variable treatment is intimately associated with the finite differencing methods used to treat the spatial variable. Another difficulty in geometries other than one dimensional planar and spherical is problems associated with a two dimensional quadrature scheme to integrate over $\vec{\Omega}$.

The P-N and S-N methods in some sense compete with each other in that both methods are rather general approximations to the equation of transfer which are capable of giving an arbitrarily small error. In neutron transport, the S-N method is more widely used than the P-N method. This is primarily because the S-N method is more easily adaptable to large digital computers. However, in certain geometries the S-N method suffers from a defect not present in the P-N method, namely the so-called "ray effect" which distorts, in a qualitative as well as a quantitative sense, the S-N solution. The origin of this effect is that the discrete rays (ordinates) may not, if they are sparse enough in number, sample an important region of the problem. Of course, the ray effect becomes less pronounced as N, the order of the approximation, is increased. In radiation-hydrodynamics, neither the P-N nor S-N treatments have been used to any extent.

H. The Monte Carlo Method

If, in radiation hydrodynamics problems, an accurate solution is desired (more accurate than is possible with any diffusion theory), the practice has been to turn to a Monte Carlo solution of the equation of transfer. This method is convenient when one has a geometrically complex configuration with little or no symmetry, or surfaces which are not naturally represented by Cartesian, cylindrical, or spherical coordinates. This method is a statistical one, in which individual photons are followed through successive collisions until the photon is either absorbed or leaks out of the system. Where the collision occurs, the result of the collision (scattering or absorption), and the direction and frequency of the photon after scattering if the collision is a scattering event, are determined by sampling from the appropriate distributions.

The basis of this sampling is the use of random numbers. Consider a random number ξ in the interval $0 \leq \xi \leq 1$. The probability for ξ to lie between ξ and $\xi + d\xi$ is, if the number is random, just proportional to the width of the interval $d\xi$, and is independent of the value of ξ. For a general distribution, with a density function $p(\xi)$, we would write

$$\text{Probability of } \xi \text{ lying between } \xi \text{ and } \xi + d\xi = p(\xi)d\xi \quad . \tag{605}$$

For random numbers, $p(\xi)$ is just a constant, say p. Since ξ must, with unit probability, lie somewhere in the interval $0 \leq \xi \leq 1$, we have

$$1 = \int_0^1 d\xi p(\xi) = p \int_0^1 d\xi = p \quad . \tag{606}$$

Thus, for a random number

$$\text{Probability of } \xi \text{ lying between } \xi \text{ and } \xi + d\xi = d\xi \quad . \tag{607}$$

As the simplest example of the Monte Carlo process, consider time independent, monofrequency, radiative transfer with isotropic scattering, a point isotropic source of photons, and a vacuum ($\Gamma = 0$) boundary condition. One would select two random numbers, ξ_1 and ξ_2, to determine the direction of a source photon. We choose the aximuthal angle ϕ as

$$\phi = 2\pi\xi_1 \quad , \tag{608}$$

and the cosine of the polar angle μ as

$$\mu = 2\xi_2 - 1 \quad . \tag{609}$$

Since $0 \leq \xi_1, \xi_2 \leq 1$, we have $0 \leq \phi \leq 2\pi$ and $-1 \leq \mu \leq 1$, the proper ranges. Further, Eqs. (608) and (609) imply uniform distributions in both ϕ and μ, which is correct for an isotropic distribution. Having chosen a direction, we need determine where the first collision occurs. Let the total cross section in this direction at a distance s from the source point be denoted by $\sigma(s)$. For a beam of photons traveling in this direction, the uncollided photon density N(s) is given by

$$N(s) = N_o \exp\left[-\int_o^s ds'\sigma(s')\right] \quad , \tag{610}$$

where N_o is the beam strength at the source point ($s = 0$). By definition of the cross section, the number of collisions occuring between s and s + ds is:

$$\begin{aligned} \text{Number of collisions} &= N(s)\sigma(s)ds \\ &= N_o\sigma(s) \exp\left[-\int_o^s ds'\sigma(s')\right]ds \quad . \end{aligned} \tag{611}$$

Hence the probability that a source photon will make a collision between s and s + ds is

$$\text{Probability} \equiv p(s)ds = \frac{\text{No. of collisions}}{N_o} , \tag{612}$$

or

$$p(s)ds = \sigma(s) \exp[- \int_o^s ds'\sigma(s')]ds . \tag{613}$$

We change variables from the variable s to a variable ξ_3, according to

$$\ell n \; \xi_3 = - \int_o^s ds'\sigma(s') . \tag{614}$$

We write

$$p(s)ds = -p(\xi_3)d\xi_3 . \tag{615}$$

[We have introduced the minus sign in Eq. (615) since ξ_3 decreases as s increases]. We wish to compute $p(\xi_3)$. From Eq. (614) we have

$$\frac{1}{\xi_3} d\xi_3 = -\sigma(s)ds , \tag{616}$$

or

$$d\xi_3 = -\sigma(s)\xi_3 ds = -\sigma(s) \exp[- \int_o^s ds'\sigma(s')]ds . \tag{617}$$

Use of this result together with Eq. (613) in Eq. (615) gives the simple result

$$p(\xi_3) = 1 . \tag{618}$$

Hence choosing the distance s from the physical distribution p(s) given by Eq. (613) is equivalent to choosing a random number in the interval $0 \leq \xi_3 \leq 1$ [by Eq. (614) this corresponds to the interval $0 \leq s < \infty$], and using Eq. (614) to determine s. That is given ξ_3, s is determined from

$$\ln(\xi_3) = -\int_0^s ds'\sigma(s') \quad . \tag{619}$$

If the cross section $\sigma(s)$ is a constant, independent of s, then Eq. (619) has the simple solution

$$s = \frac{1}{\sigma} \ln (1/\xi_3) \quad . \tag{620}$$

Once the location of the first collision is determined, a further random number ξ_4 is used to determine if the collision is scattering or absorption. If ξ_4 lies in the range

$$0 \leq \xi_4 \leq \tilde{\omega} = \frac{\sigma_s}{\sigma} \quad , \tag{621}$$

the collision would be taken as a scattering collision, and if ξ_4 lies in the range

$$\tilde{\omega} < \xi_4 \leq 1 \quad , \tag{622}$$

the collision would be taken as absorption. If the collision were determined to be a scattering event, further random numbers would be used to determine the direction of the photon after scattering, as in Eqs. (608) and (609), and its distance to the next collision, as in Eq. (619). This process would be continued until the photon is lost, either by absorption or leakage from the system. Then another photon, generated with new random numbers, would be started from the source point. In a real, multifrequency, problem, sampling from the appropriate source and scattering frequency distributions would also be employed. Additional complications in real problems include time dependence, distributed sources, photons entering the system through the boundaries ($\Gamma \neq 0$) and non-isotropic scattering and sources.

In the Monte Carlo method, the results of each collision are tallied, and from this information the specific intensity $I(\vec{r},\nu,\vec{\Omega},t)$ can be constructed. If one ran an infinite number of photons, the result for $I(\vec{r},\nu,\vec{\Omega},t)$ would be an exact solution to the equation of transfer. In practice, of course, one runs only a finite number of photons, and this introduces a statistical fluctuation into the answer. Monte Carlo codes not only produce an answer, but generally give variance information which gives some idea of the statistical uncertainty of the answer. A myriad of schemes are in use to reduce this statistical variance. These generally go under the name of biasing.

I. The Integral (Formal) Solution Method.

One could envision solving the equation of transfer by employing the integral formulation, which is just following photons along their characteristics. This is conceptually very simple. One would pass a multitude of rays through the system, in sufficient number to adequately sample all spatial regions and angular directions. One would then evaluate the formal solution to the equation of transfer. In steady state, this is just Eq. (116), i.e.,

$$I(\vec{r},\vec{\Omega}) = \Gamma(\vec{r}_s,\vec{\Omega}) \exp\left[-\int_0^{|\vec{r}-\vec{r}_s|} ds''\sigma(\vec{r} - s''\vec{\Omega})\right]$$

$$+ \int_0^{|\vec{r}-\vec{r}_s|} ds'Q(\vec{r} - s'\vec{\Omega},\vec{\Omega}) \exp\left[-\int_0^{s'} ds''\sigma(\vec{r} - s''\vec{\Omega})\right] \quad , (623)$$

where $Q(\vec{r},\nu,\vec{\Omega})$ is the total (emission plus scattering) source given by Eq. (109), i.e.,

$$Q(\vec{r},\nu,\vec{\Omega}) = S(\vec{r},\nu)$$

$$+ \int_0^{\infty} d\nu' \int_{4\pi} d\vec{\Omega}' \frac{\nu}{\nu'} \sigma_s(\vec{r},\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}')I(\vec{r},\nu',\vec{\Omega}') \quad . (624)$$

The scattering would have to be handled iteratively. In time dependent problems, the formal solution to be evaluated is given by Eq. (118). In this case, scattering could be treated explicitly in time, or iterated for increased accuracy and stability.

J. The Multigroup Method

Thus far we have discussed various approximate methods for dealing with the angular variable. We conclude our discussion of approximation methods by considering the frequency variable. The generally used procedure for handling the frequency variable in the equation of transfer is the multigroup method, which really amounts to a discretization of the frequency variable. Rather than treating the frequency as a continuous variable, one assigns a given photon to one of G frequency groups, and all photons within a given group are treated the same, assigning average properties, such as the absorption coefficient, to these photons.

To introduce the multigroup method, we consider the equation of transfer with no scattering, and with an absorption coefficient σ_a which is independent of frequency. We then have

$$\frac{1}{c}\,\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \sigma_a\left[B(\nu) - I(\nu,\vec{\Omega})\right] \quad . \tag{625}$$

According to Eqs. (21) through (23), the radiative energy density, flux, and pressure tensor, the quantities of particular interest in radiation hydrodynamic problems, all involve integrals of the specific intensity over the frequency variable. Thus it is reasonable to integrate the equation of transfer, Eq. (625), over frequency. We find, since σ_a is independent of frequency,

$$\frac{1}{c}\,\frac{\partial I(\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\vec{\Omega}) = \sigma_a\left[\frac{ac}{4\pi}\,T^4 - I(\vec{\Omega})\right] \quad , \tag{626}$$

where we have defined

$$I(\vec{r},\vec{\Omega},t) = \int_0^\infty d\nu I(\vec{r},\nu,\vec{\Omega},t) \quad . \tag{627}$$

Equation (626) is exact, and is referred to as the grey, or one group, equation of transfer since all photons are treated together in a single frequency group extending from $\nu = 0$ to $\nu = \infty$.

Let us again consider the equation of transfer with no scattering but with an absorption coefficient which depends upon photon frequency, i.e.,

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \sigma_a(\nu)\left[B(\nu) - I(\nu,\vec{\Omega})\right] \quad . \tag{628}$$

The grey equation of transfer associated with Eq. (628) is generally taken as, in analogy to Eq. (626),

$$\frac{1}{c}\frac{\partial I(\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\vec{\Omega}) = \overline{\sigma}_a\left[\frac{ac}{4\pi}T^4 - I(\vec{\Omega})\right] \quad , \tag{629}$$

where $I(\vec{\Omega}) = I(\vec{r},\vec{\Omega},t)$ is again defined by Eq. (627). Here $\overline{\sigma}_a$ is some kind of mean absorption coefficient averaged over frequency. If $\overline{\sigma}_a$ is allowed to be a function of space and time only, as is generally the case in practice, Eq. (629) is an approximate equation for $I(\vec{r},\vec{\Omega},t)$ no matter what choice is made for $\overline{\sigma}_a$. This is easily seen by integrating Eq. (628) over all frequency. One indeed finds a result like Eq. (629), but with the important difference that $\overline{\sigma}_a$ is a function of $\vec{\Omega}$ as well as $\vec{r}$ and t. In fact, $\overline{\sigma}_a$ must be defined as

$$\overline{\sigma}_s(\vec{r},\vec{\Omega},t) = \frac{\int_0^\infty d\nu\ \sigma_a(\nu)\left[B(\nu,t) - I(\vec{r},\nu,\vec{\Omega},t)\right]}{\int_0^\infty d\nu\left[B(\nu,T) - I(\vec{r},\nu,\vec{\Omega},t)\right]} \quad , \tag{630}$$

for Eq. (629) to be an exact one group equation. Of course, for Eq. (629) to be useful, one must know $\bar{\sigma}_a$ which, according to Eq. (630), involves the unknown specific intensity. [If the specific intensity were known, the radiative transfer problem would be solved.] We return to this point shortly. We re-emphasize at this time, however, that in practice $\bar{\sigma}_a$ is universally chosen as independent of $\vec{\Omega}$, which in general implies an approximation.

With this introduction, we now consider the equation of transfer with scattering (however, for simplicity we neglect induced processes) and construct the more general multigroup equations. We shall discuss the utility of the multigroup equations, even though one may only be interested in one group results (the specific intensity integrated over all frequency), following our derivation of the multigroup equations. The equation of transfer we consider is

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega})$$

$$= \sigma_a(\nu)\left[B(\nu) - I(\nu,\vec{\Omega})\right] - \sigma_s(\nu)I(\nu,\vec{\Omega})$$

$$+ \int_{4\pi} d\vec{\Omega}' \int_0^\infty d\nu' \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})I(\nu',\vec{\Omega}') \quad . \qquad (631)$$

We divide the frequency range into G groups with boundaries $\nu = \nu_0 = 0, \nu_1, \nu_2, \ldots \nu_{G-1}, \nu_G = \infty$ and define the g^{th} group specific intensity as

$$I_g(\vec{r},\vec{\Omega},t) = \int_{\nu_{g-1}}^{\nu_g} d\nu I(\vec{r},\nu,\vec{\Omega},t) \ , \quad 1 \le g \le G \ . \qquad (632)$$

Integration of Eq. (631) over the g^{th} group yields

$$\frac{1}{c}\frac{\partial I_g(\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I_g(\vec{\Omega})$$

$$= \sigma_{ag}(\vec{\Omega})\left[b_g \frac{ac}{4\pi}T^4 - I_g(\vec{\Omega})\right] - \sigma_{sg}(\vec{\Omega})I_g(\vec{\Omega})$$

$$+ \int_{4\pi} d\vec{\Omega}' \sum_{g'=1}^{G} \sigma_{s,g'\to g}(\vec{\Omega},\vec{\Omega}')I_g(\vec{\Omega}') \quad ,$$

$$1 \le g \le G \quad , \qquad (633)$$

where b_g is defined as that fraction of $acT^4/4\pi$ which lies within the g^{th} group, i.e.,

$$b_g = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu B(\nu)}{\int_0^\infty d\nu B(\nu)} = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu B(\nu)}{acT^4/4\pi} \quad . \qquad (634)$$

Equation (633) is exact providing we define the g^{th} group interaction coefficients as

$$\sigma_{ag}(\vec{\Omega}) = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu\sigma_a(\nu)\left[B(\nu) - I(\nu,\vec{\Omega})\right]}{\int_{\nu_{g-1}}^{\nu_g} d\nu\left[B(\nu) - I(\nu,\vec{\Omega})\right]} \quad , \qquad (635)$$

$$\sigma_{sg}(\vec{\Omega}) = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu\sigma_s(\nu)I(\nu,\vec{\Omega})}{\int_{\nu_{g-1}}^{\nu_g} d\nu I(\nu,\vec{\Omega})} \quad , \qquad (636)$$

$$\sigma_{s,g'\to g}(\vec{\Omega},\vec{\Omega}') = \frac{\int_{\nu_{g'-1}}^{\nu_{g'}} d\nu' \int_{\nu_{g-1}}^{\nu_g} d\nu \frac{\nu}{\nu'} \sigma_s(\nu'\to\nu,\vec{\Omega}\cdot\vec{\Omega}') I(\nu',\vec{\Omega}')}{\int_{\nu_{g'-1}}^{\nu_{g'}} d\nu' I(\nu',\vec{\Omega}')} . \quad (637)$$

The G equations given by Eq. (633) represent the general form of the multigroup equations. These equations are coupled through the scattering interaction and must be solved simultaneously.

For the multigroup equations to be useful, one must be able to compute or estimate the group constants defined by Eqs. (635) through (637). An exact calculation of these constants involves a complete knowledge of the specific intensity which, of course, is unknown. The underlying assumption in the multigroup method is that these group constants, since they are homogeneous functionals of the specific intensity, are relatively insensitive to the weighting function $I(\vec{r},\nu,\vec{\Omega},t)$ [or B - I in the case of Eq. (635)] used in computing these averages over frequency. Hence one hopes that a relatively crude estimate for the specific intensity will lead to reasonably accurate group constants. As the group width becomes smaller, of course, the group constants become increasingly less dependent upon the estimate made for $I(\vec{r},\nu,\vec{\Omega},t)$. This is the reason that a multigroup formulation of the frequency variable is preferable to a one group, or grey, treatment even though the ultimate goal may be to compute one group results (energy density, flux, and pressure tensor).

To evaluate the group constants involving the scattering kernel, Eqs. (636) and (637), a reasonable choice for $I(\vec{r},\nu,\vec{\Omega},t)$ would be the Planck function at the local material temperature. This ensures correctness as one approaches equilibrium, i.e., at thermodynamic equilibrium the specific intensity is in fact given by the Planck function. Away from equilibrium, the only justification for the use of the Planck function is that the scattering cross section is a relatively smooth function of frequency, and hence the choice of the weighting function in these group constants is not crucial as long as a reasonable function is used. In the case of σ_{ag}, the average absorption coefficient, more care should be taken. Absorption coefficients encountered in practice are generally complex and widely varying functions of frequency, and the use of different weighting functions can lead to quite different results for σ_{ag}. In practice, σ_{ag} is generally taken as either a group Rosseland or group Planck mean.

The Rosseland mean, similar to that introduced earlier [see Eq. (485)], follows from the assumption that the specific intensity is given by the equilibrium diffusion approximation,

$$B(\nu,T) - I(\vec{r},\nu,\vec{\Omega},t) = \frac{1}{\sigma(\vec{r},\nu,t)} \frac{\partial B(\nu,T)}{\partial T} \vec{\Omega}\cdot\vec{\nabla}T(\vec{r},t) \quad , \tag{638}$$

where $\sigma = \sigma_a + \sigma_s$. Use of this result in Eq. (635) yields a Rosseland-like result

$$\sigma_{ag} = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu \, \frac{\sigma_a(\nu)}{\sigma(\nu)} \frac{\partial B(\nu,T)}{\partial T}}{\int_{\nu_{g-1}}^{\nu_g} d\nu \, \frac{1}{\sigma(\nu)} \frac{\partial B(\nu,T)}{\partial T}} \quad , \tag{639}$$

The Planck mean is appropriate in the case of time independent radiative transfer in an optically thin, emission dominated, system. Once can easily show that in an optically thin system, the specific intensity I is small compared to the Planck function B. That is, in this case we have

$$B(\nu,T) - I(\vec{r},\nu,\vec{\Omega}) \simeq B(\nu,T) \quad , \tag{640}$$

and Eq. (635) then gives

$$\sigma_{ag} = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu \sigma_a(\nu) B(\nu,T)}{\int_{\nu_{g-1}}^{\nu_g} d\nu B(\nu,T)} \quad . \tag{641}$$

This type of average is generally referred to as the g^{th} group Planck mean, or Planck averaged, absorption coefficient.

As the above discussion indicates, the use of the Rosseland and Planck mean absorption coefficients is only strictly appropriate in limiting circumstances. Nevertheless, one or the other mean is generally used in the multigroup method. For realistic absorprion coefficients, these two means can differ by an order of magnitude or more, and thus the results of the multigroup method can vary widely depending upon which mean is used. In truth, neither mean is correct in general. For most problems of radiation hydrodynamics, experience indicates that the use of the Rosseland mean is the more accurate of the two. In fact, Eq. (639) is often approximated by arguing the $\partial B/\partial T$ is sufficiently slowly varying over a group, so that Eq. (639) can be replaced by

$$\sigma_{ag} = \frac{\int_{\nu_{g-1}}^{\nu_g} d\nu \, \frac{\sigma_a(\nu)}{\sigma(\nu)}}{\int_{\nu_{g-1}}^{\nu_g} d\nu \, \frac{1}{\sigma(\nu)}} \quad , \tag{642}$$

For completeness, we give expressions for the radiative energy density, radiative flux, and radiative pressure tensor in the multigroup approximation. From Eqs. (21) through (23) and Eq. (632) we have

$$E(\vec{r},t) = \frac{1}{c} \sum_{g=1}^{G} \int_{4\pi} d\vec{\Omega} I_g(\vec{r},\vec{\Omega},t) \quad , \tag{643}$$

$$\vec{F}(\vec{r},t) = \sum_{g=1}^{G} \int_{4\pi} d\vec{\Omega}\vec{\Omega} I_g(\vec{r},\vec{\Omega},t) \quad , \tag{644}$$

$$\vec{\vec{P}}(\vec{r},t) = \frac{1}{c} \sum_{g=1}^{G} \int_{4\pi} d\vec{\Omega}\vec{\Omega}\vec{\Omega} I_g(\vec{r},\vec{\Omega},t) \quad . \tag{645}$$

The multigroup scheme is combined with one of the angular approximations previously discussed, together with some kind of treatment in space and time (finite difference, Monte Carlo, formal solution, etc.) to yield a practical calculational scheme for radiation hydrodynamic problems.

IV. THE INTERACTION OF THE RADIATION FIELD WITH MATTER

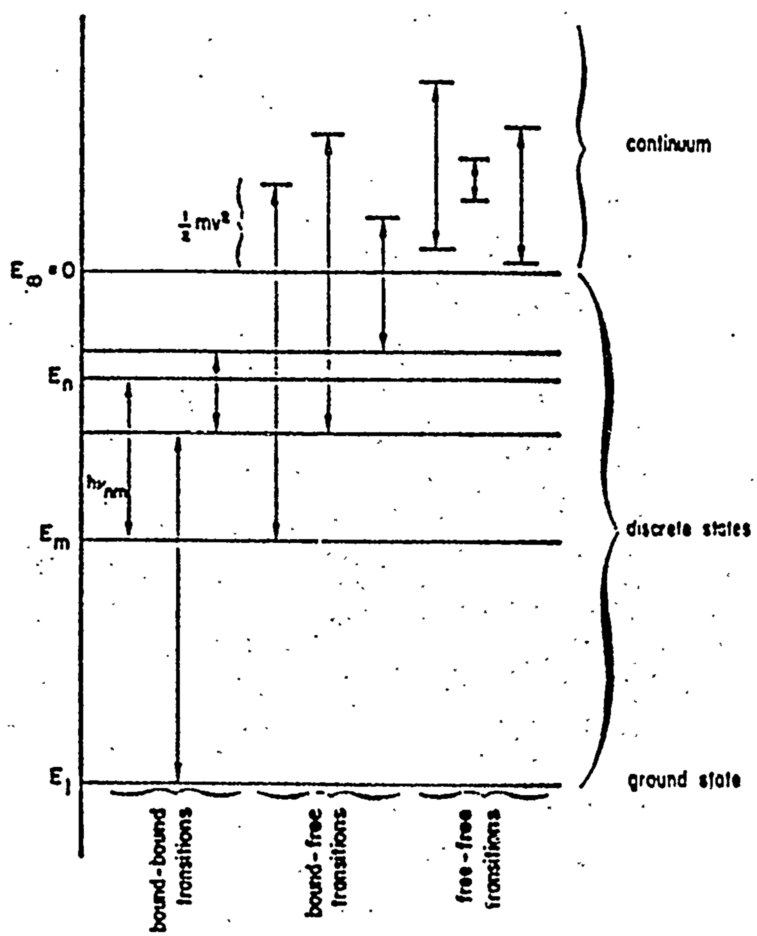
Thus far we have discussed the equation of radiative transfer and the equations of radiation hydrodynamics in various forms. For the most part, all of these results followed from the simple notion of conservation of photons, mass, momentum, and energy on a macroscopic level. The underlying physics of radiative transfer is contained in the absorption coefficient $\sigma_a(\nu)$, the scattering kernel $\sigma_s(\nu'\rightarrow\nu,\vec{\Omega}\cdot\vec{\Omega}')$ and the spontaneous emission source $S(\nu)$. We now give a very brief discussion of this aspect of radiative transfer. The only topic we treat in any detail is Compton scattering from free electrons.

A. Absorption and Source

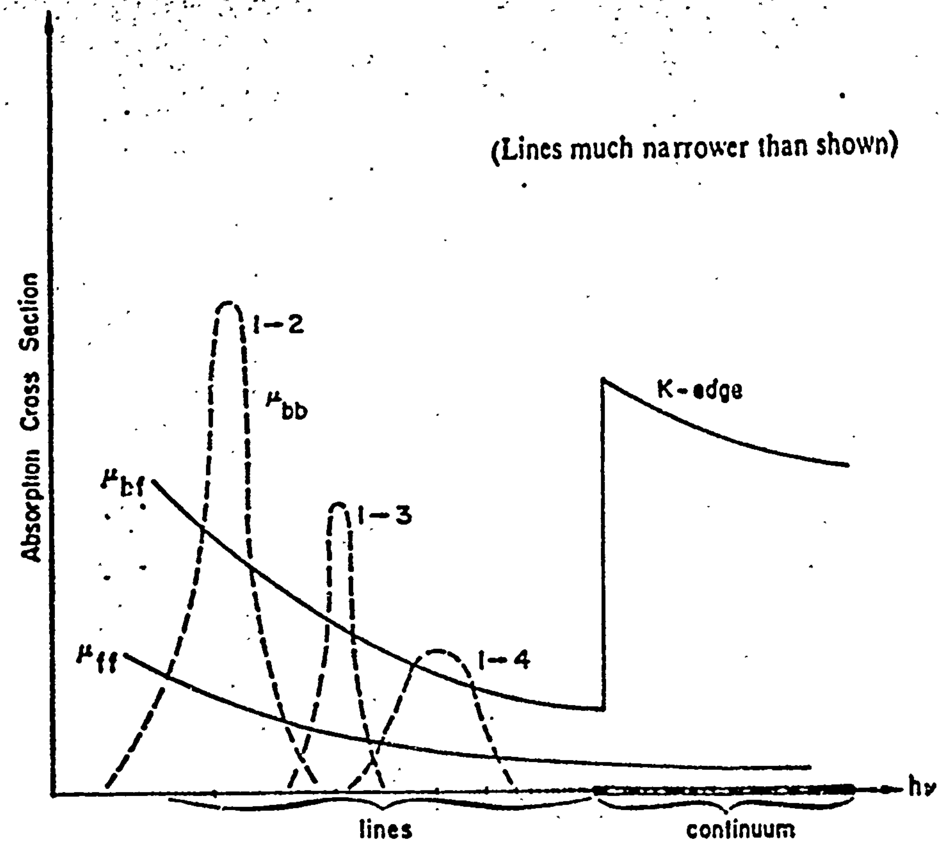
The calculation of the absorption (and scattering) coefficient and the source function involves two conceptually distinct steps. In the first place, assuming LTE, given the temperature, density, and atomic composition of a plasma, one requires a quantitative statement concerning the population of the various ionic species present. In addition, for each ionic species one needs the population of each quantum energy state.

Secondly, given the populations one requires the probability that a photon will induce a transition from one quantum state to another. This requires a study of atomic and molecular processes together with the quantum theory of radiation (quantum electrodynamics).

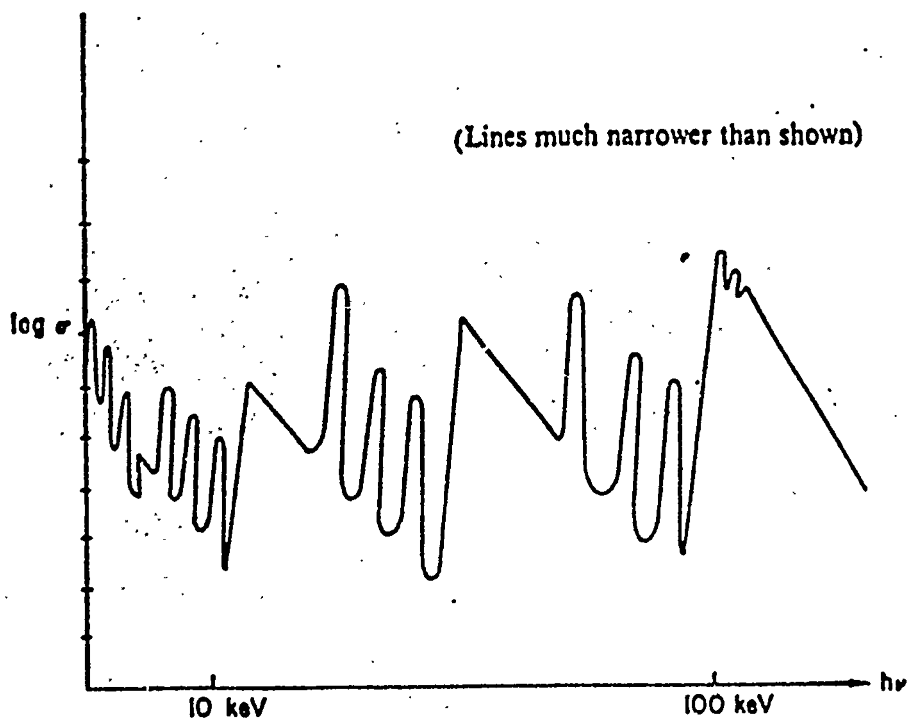
The mechanisms of absorption of radiation by matter are bound-bound (line) absorption, bound-free (photoelectric absorption, and free-free (continuum) absorption. In line absorption, an electron in a bound state is excited to another bound state of higher energy by the absorption of a photon. The frequency of the absorption line is given by Bohr's relationship $h\nu_{nm} = E_n - E_m$, where E_n and E_m are the higher and lower energy states, respectively. In photoelectric absorption, the electron is ejected from the atom or ion and goes into one of the continuum of free energy states. Photoelectric absorption occurs whenever the energy of the incident photon is greater than the binding energies of the electrons of the atoms or ions. In free-free absorption, an electron in a free state makes a transition to another free state of higher energy with the absorption of a photon. This is shown schematically below.



Atomic energy levels and transitions.



Comparison of absorption processes.



Total absorption coefficient.

B. Scattering

The most important scattering process in radiation hydrodynamic problems is scattering from free electrons, called Compton scattering. In the low frequency limit, i.e., when

$$\frac{h\nu}{m_o c^2} << 1 \quad , \tag{648}$$

where $m_o c^2 = 0.511$ MeV (m_o is the rest electron mass), Compton scattering limits to Thomson scattering, given by

$$\sigma_s(\nu \to \nu', \xi) = N \frac{r_o^2}{2} (1 + \xi^2) \delta(\nu - \nu') \quad , \tag{649}$$

where $\xi = \vec{\Omega} \cdot \vec{\Omega}'$, N is the electron density, and r_o is the classical electron radius

$$r_o = \frac{e^2}{m_o c^2} \quad . \tag{650}$$

Integration of Eq. (649) over all ν' and solid angle gives the scattering cross section as:

$$\sigma_s = \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_s(\nu \to \nu', \xi) = \frac{8\pi}{3} N r_o^2 = N(\frac{2}{3} \text{ barns}) \quad . \tag{651}$$

We note three characteristics of Thomson scattering:

1. It is coherent (no frequency change upon scattering);
2. It is symmetric in the forward and backward hemispheres of the scattering angle;
3. The scattering cross section is independent of frequency.

Equation (651) is the classical result of scattering from free electrons after averaging over polarization states.

The quantum mechanical result for the scattering of photons by free electrons is given by the Klein-Nishina formula. As in the classical result, the scattering kernel actually depends upon the state of polarization of the incident photon. If one averages over polarization states (assumes natural, unpolarized, light), the scattering kernel is given by, for electrons at rest,

$$\sigma_a(\nu\to\nu',\xi) = N\frac{r_o^2}{2}\frac{1+\xi^2}{[1+\gamma(1-\xi)]^2}\cdot$$

$$\cdot\left\{1+\frac{\gamma^2(1-\xi)^2}{(1+\xi^2)[1+\gamma(1-\xi)]}\right\}\delta\left(\nu'-\frac{\nu}{1+\gamma(1-\xi)}\right)\quad,\qquad(652)$$

where γ is a dimensionless frequency

$$\gamma = \frac{h\nu}{m_o c^2}\ .\qquad(653)$$

The Dirac delta function in Eq. (652) states that, given an initial photon frequency, the scattering angle and final photon energy are correlated. This correlation results from simple conservation of energy and momentum in the scattering process. We see from the argument of the delta function that $h\nu'$, the final energy, is always less than $h\nu$, the initial energy. This energy difference is the recoil energy given to the free electron.

Another widely seen form of the Klein-Nishina formula follows from Eq. (652) by changing the delta function from one in ν' to one in ξ. Suppressing the algebraic detail, we find

$$\sigma_s(\nu\to\nu',\xi) = N\frac{r_o^2}{2}\frac{1}{\gamma\nu}\left[\frac{\gamma}{\gamma'} + \frac{\gamma'}{\gamma} + 2\left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right) + \left(\frac{1}{\gamma} - \frac{1}{\gamma'}\right)^2\right] \cdot$$

$$\cdot\ \delta\left(\xi - 1 + \frac{1}{\gamma'} - \frac{1}{\gamma}\right) \quad , \tag{654}$$

with γ given by Eq. (653) and

$$\gamma' = \frac{h\nu'}{m_o c^2} \quad . \tag{655}$$

A third form for $\sigma_s(\nu\to\nu',\xi)$ follows very simply from Eq. (655). This is

$$\sigma_s(\nu\to\nu',\xi) = N\frac{r_o^2}{2}\frac{1}{\gamma\nu}\left[1 + \xi^2 + \gamma\gamma'(1 - \xi)^2\right]$$

$$\cdot\ \delta\left(\xi - 1 + \frac{1}{\gamma'} - \frac{1}{\gamma}\right) \quad . \tag{656}$$

For low incident energies, i.e., $\gamma \ll 1$, it is sensible to expand Eq. (652) in powers of γ. To second order, we have

$$\sigma_s(\nu\to\nu',\xi) = N\frac{r_o^2}{2}(1 + \xi^2)\left[1 - 2\gamma(1-\xi) + \gamma^2\frac{(1-\xi)^2(4+3\xi^2)}{1 + \xi^2}\right] \cdot$$

$$\cdot\ \delta\left(\nu' - \nu\left[1 - \gamma(1 - \xi) + \gamma^2(1 - \xi)^2\right]\right) \quad . \tag{657}$$

Setting $\gamma = 0$ in Eq. (657), we obtain

$$\sigma_s(\nu\to\nu',\xi) = N\frac{r_o^2}{2}(1 + \xi^2)\delta(\nu' - \nu) \quad , \tag{658}$$

which is just the classical Thomson scattering kernel [see Eq. (649)]. Integration of Eq. (652) over all $\vec{\Omega}'$ and ν' gives the scattering cross section $\sigma_s(\nu)$. We find

$$\sigma_s(\nu) = \frac{3N\mu_{Th}}{4}\{(\frac{1+\gamma}{\gamma^3})[\frac{2\gamma(1+\gamma)}{1+2\gamma} - \ln(1+2\gamma)]$$

$$+ \frac{1}{2\gamma}\ln(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2}\} \quad , \tag{659}$$

where μ_{Th} is the Thomson microscopic scattering cross section

$$\mu_{Th} = \frac{8\pi}{3} r_o^2 = \frac{8\pi e^4}{3m_o^2c^4} \quad . \tag{660}$$

Correct to second order in γ, Eq. (659) gives

$$\sigma_s(\nu) = N\mu_{Th}\left(1 - 2\gamma + \frac{26}{5}\gamma^2\right) \quad . \tag{661}$$

For small γ, Eq. (661) shows that the Compton scattering cross section is smaller than the classical Thomson value. It can be shown from Eq. (659) that this inequality is true for all γ (i.e., for all incident photon energies).

C. Inverse Compton Scattering

The Klein-Nishina formula describes Compton scattering from free electrons at rest, and exhibits the characteristic that photons cannot gain energy upon scattering (this is the so-called Compton shift). If the interaction is between a photon and a moving electron, however, the electron can impart some or all of its energy to the photon and increase the photon's frequency upon scattering. Such an event is often referred to as inverse Compton scattering. We now derive the scattering kernel in the case of photon scattering from a relativistic Maxwellian gas of free, non-degenerate, electrons. The elements of the derivation, however, are applicable to the more general case of photon scattering from an arbitrary distribution of moving particles.

Before proceeding analytically, it is worthwhile to briefly discuss the nature of the scattering kernel expected in this case. The kernel will have three rather distinct characteristics. In the first place, a photon will, upon scattering, have its wavelength increased due to the usual Compton shift associated with scattering from an electron at rest. Secondly, it will undergo broadening due to the classical Doppler effect of scattering from a distribution of moving electrons. Finally, there will be a reduction in the wavelength upon scattering due to the relativistic effect that the photon density will appear more intense to an electron moving toward the photon than away from it. This last effect, the blue shift, is needed to "balance" the Compton red shift, for, if black body radiation at a certain temperature scatters from a Maxwellian gas of free electrons at the same temperature, the scattered radiation must have the same distribution in wavelength as the incident radiation.

Since the Maxwellian distribution is the thermodynamic equilibrium distribution for the electrons, the scattering kernel $\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})$ must also satisfy the detailed balance condition. This condition states that in complete thermodynamic equilibrium the number of photons which scatter from $d\nu'd\vec{\Omega}'$ about $\nu',\vec{\Omega}'$ to $d\nu d\vec{\Omega}$ about $\nu,\vec{\Omega}$ must equal the number scattered from $d\nu d\vec{\Omega}$ to $d\nu'd\vec{\Omega}'$. Quantitatively, this condition takes the form

$$[1 + c^2B(\nu)/2h\nu^3]\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})B(\nu')/h\nu'$$

$$= [1 + c^2B(\nu')/2h\nu'^3]\sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}')B(\nu)/h\nu \quad , \qquad (662)$$

where $B(\nu)$ is the Planck function given by

$$B(\nu) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1} \quad . \tag{663}$$

Equation (663) relates the scattering kernel, at a given scattering angle, to the kernel with the frequency variables ν and ν' interchanged, at the same angle. Explicit use of Eq. (663) in Eq. (662) yields

$$\sigma_s(\nu'\to\nu,\vec{\Omega}'\cdot\vec{\Omega})W(\nu')/h\nu' = \sigma_s(\nu\to\nu',\vec{\Omega}\cdot\vec{\Omega}')W(\nu)/h\nu \quad , \tag{664}$$

where $W(\nu)$ is the Wien approximation to the Planck function. Aside from a normalization, we have

$$W(\nu) = \nu^3 e^{-h\nu/kT} \quad . \tag{665}$$

This result can be interpreted as the detailed balance condition in the absence of induced scattering and shows that the neglect of these induced terms in the scattering description leads to the Wien law, rather than the Planck function, as the equilibrium distribution of the scattering operator.

To compute the scattering kernel we have just discussed, we consider a frame of reference in which a group of electrons is at rest. We call this the e frame and subscript all quantities in this frame with an e. We take these electrons to have a density N_e in this frame. If the unadorned frame moves with velocity $-\vec{v}$ with respect to the e frame (so that, as observed from the unadorned frame, the electrons have velocity $\vec{v}$) we have from Eq. (240),

$$\sigma_s(\nu\to\nu',\vec{\Omega}\to\vec{\Omega}') = \frac{D}{D'}\,\sigma_{se}(\nu_e\to\nu_e',\vec{\Omega}_e\to\vec{\Omega}_e') \quad , \tag{666}$$

where

$$D = 1 - \vec{\Omega}\cdot\vec{v}/c \quad ; \quad D' = 1 - \vec{\Omega}'\cdot\vec{v}/c \quad . \tag{667}$$

[Note the change of sign in Eq. (667) as compared to Eqs. (229) and (230). This is because here the unadorned frame moves with velocity $-\vec{v}$ with respect to the e frame. The e frame here is to be identified with the zero frame in Eqs. (228) through (241).] In the e frame, the scattering kernel $\sigma_{se}(\nu_e \to \nu_e', \vec{\Omega}_e \to \vec{\Omega}_e')$ is just the Klein-Nishina formula, Eq. (656), with $\xi = \vec{\Omega} \cdot \vec{\Omega}'$ and all quantities subscripted with an e. The independent variables, frequency and angle, transform as, from Eqs. (231) and (234)

$$\nu_e = \lambda D \nu \quad , \tag{668}$$

$$\nu_e' = \lambda D \nu' \quad , \tag{669}$$

$$1 - \vec{\Omega}_e \cdot \vec{\Omega}_e' = (1 - \vec{\Omega} \cdot \vec{\Omega}')/\lambda^2 D D' \quad , \tag{670}$$

with D and D' given by Eq. (667) and

$$\lambda = (1 - v^2/c^2)^{-1/2} \quad . \tag{671}$$

Also, due to the Lorentz contraction, the electron density in the unadorned frame is given by

$$N = \lambda N_e \quad , \tag{672}$$

with λ given by Eq. (671). Combining all of these results, we obtain

$$\sigma_s(\nu \to \nu', \vec{\Omega} \to \vec{\Omega}') = \frac{N r_o^2}{2\lambda\gamma\nu} \delta\left(\xi - 1 + \frac{\lambda D}{\gamma'} - \frac{\lambda D'}{\gamma}\right) \cdot$$

$$\cdot \left\{1 + \left[1 - \frac{(1 - \xi)}{\lambda^2 D D'}\right]^2 + \frac{\gamma\gamma'(1 - \xi)^2}{\lambda^2 D D'}\right\} \quad , \tag{673}$$

where $\xi = \vec{\Omega}\cdot\vec{\Omega}'$.

Equation (673) is the scattering kernel corresponding to all electrons moving with a velocity $\vec{v}$. To account for the fact that the electrons have a velocity distribution (namely, a Maxwellian) as seen by the observer in the unadorned frame, we replace N in Eq. (673) by

$$N \to Nf(v)d\vec{v} \quad , \tag{674}$$

where f(v) is the isotropic Maxwellian distribution function normalized according to

$$\int d\vec{v}f(v) = 4\pi \int_0^c dvv^2 f(v) = 1 \quad , \tag{675}$$

and integrate the resulting expression over all $\vec{v}$. [We note that since the Maxwellian is isotropic, the distribution function depends only upon the speed v rather than the velocity $\vec{v}$.]. This gives

$$\sigma_s(\nu\to\nu',\xi) = \frac{Nr_o^2}{2\gamma\nu} \int d\vec{v}f(v)\, \frac{1}{\lambda}\, \delta\left(\xi - 1 + \frac{\lambda D}{\gamma'} - \frac{\lambda D'}{\gamma}\right) \cdot$$

$$\cdot \left\{1 + \left[1 - \frac{(1-\xi)}{\lambda^2 DD'}\right]^2 + \frac{\gamma\gamma'(1-\xi)^2}{\lambda^2 DD'}\right\} \quad . \tag{676}$$

as the scattering kernel describing scattering from free electrons of density N, and with an isotropic velocity distribution f(v). In writing the left hand side of Eq. (676), we have anticipated the fact that this kernel will depend only upon $\xi = \vec{\Omega}\cdot\vec{\Omega}'$, rather than $\vec{\Omega}$ and $\vec{\Omega}'$ separately, since the electron distribution is isotropic.

To compute the (relativistic) Maxwellian distribution of free electrons, we use the fact that $\psi(\vec{p})$, the distribution function per unit momentum, is given by the Maxwell-Boltzmann distribution

$$\psi(\vec{p}) = Ce^{-E/kT} \quad , \tag{677}$$

where C is a normalization constant. To transform from $\psi(\vec{p})$ to f(v), the distribution function per unity velocity, we need introduce the Jacobian of the transformation from $\vec{p}$ to $\vec{v}$, i.e.,

$$J(\vec{p};\vec{v}) = \frac{p^2}{v^2}\frac{dp}{dv} \quad . \tag{678}$$

Since

$$p = m_o \lambda v \quad , \tag{679}$$

and

$$E = m_o c^2 \lambda \quad , \tag{680}$$

we find

$$f(v) = C'\lambda^5 \exp - (m_o c^2 \lambda/kT) \quad , \tag{681}$$

where C' is another constant and as before, λ is the Lorentz factor given by Eq. (671). If we demand that f(v) be normalized to have an integral of unity according to Eq. (675), the constant C' is easily evaluated. Our final result for the normalized Maxwellian distribution is then

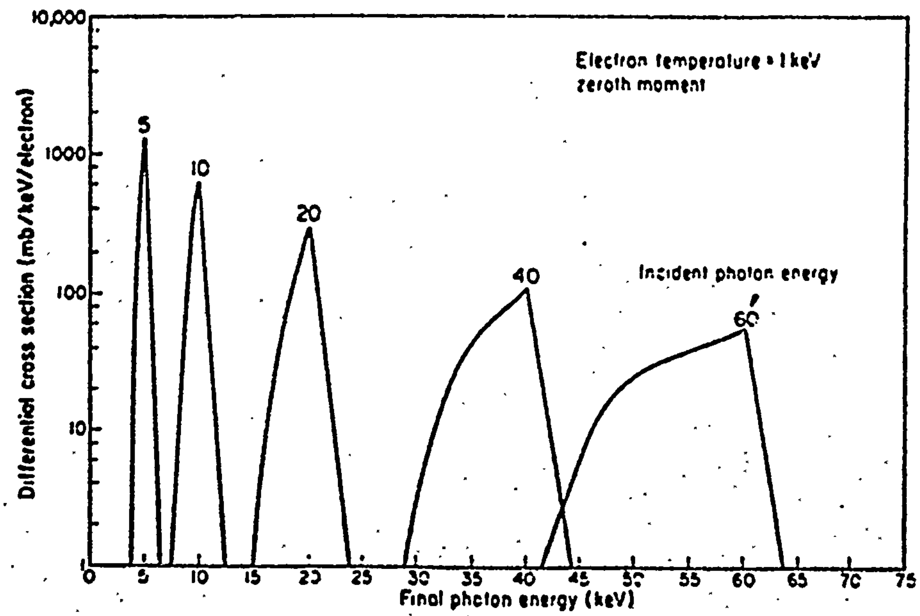
$$f(v) = \frac{m_o \lambda^5 e^{-m_o c^2 \lambda/kT}}{4\pi ckTK_2(m_o c^2/kT)} \quad , \tag{682}$$

where $K_2(z)$ is the modified Bessel function of the second kind of order two, and m_0 is the rest mass of the electron.

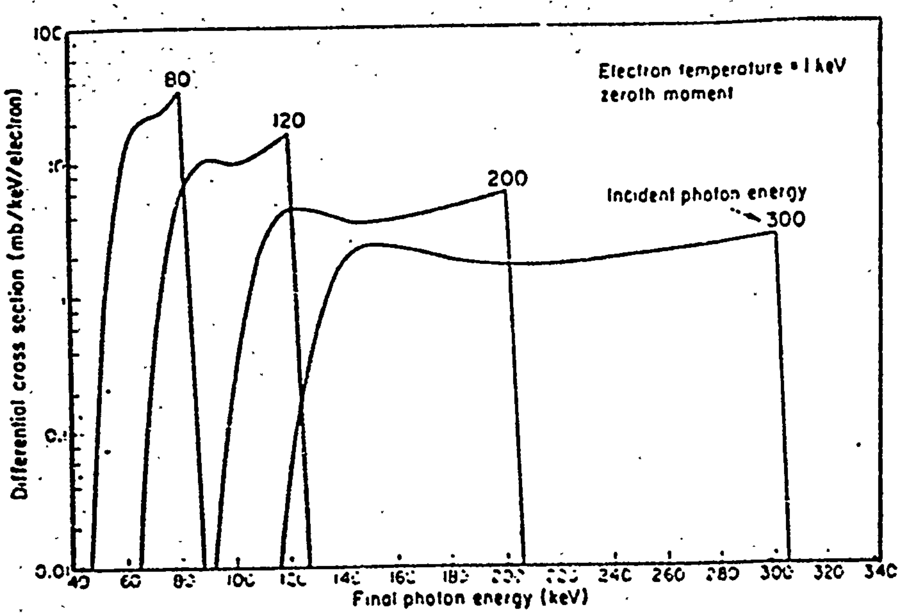
Performing the triple integral indicated in Eq. (676), with $f(v)$ given by Eq. (682), gives the scattering kernel for Compton and inverse Compton scattering. One of these three integrals can be performed analytically because of the presence of the delta function. The remaining two integrals must be evaluated numerically. The Legendre moments of this scattering kernel are defined in the usual way as

$$\sigma_{sn}(\nu\to\nu') = 2\pi \int_{-1}^{1} d\xi P_n(\xi)\sigma_s(\nu\to\nu',\xi) \quad . \tag{683}$$

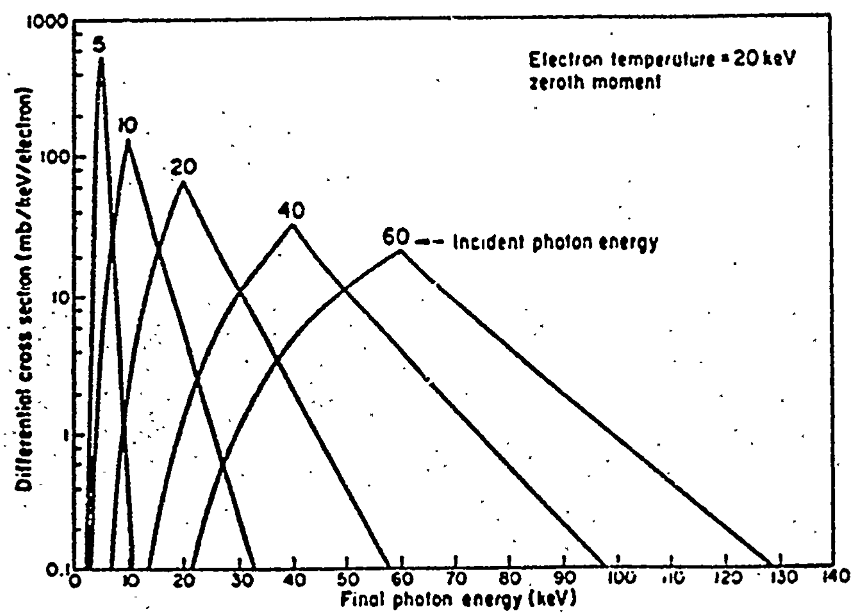
Typical results for these moments are shown in the following figures:



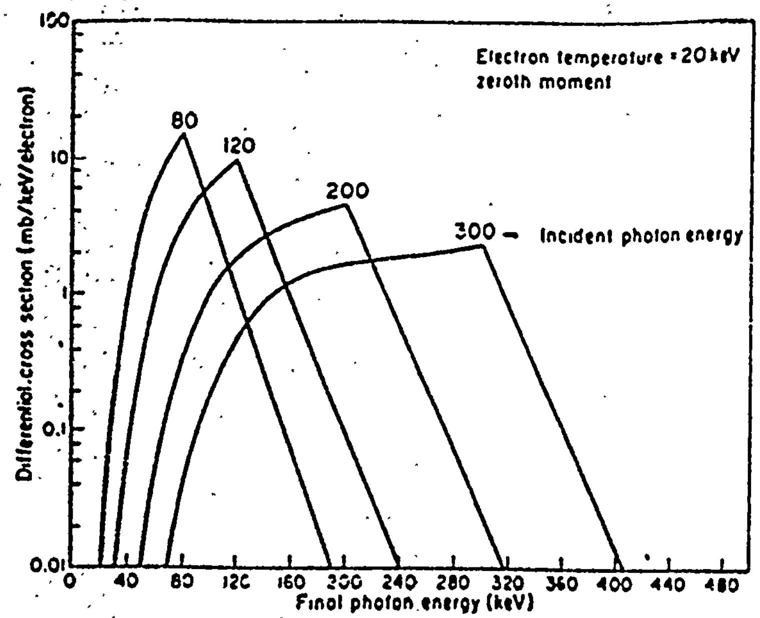
The differential scattering cross section: $n = 0$; $T = 1$.



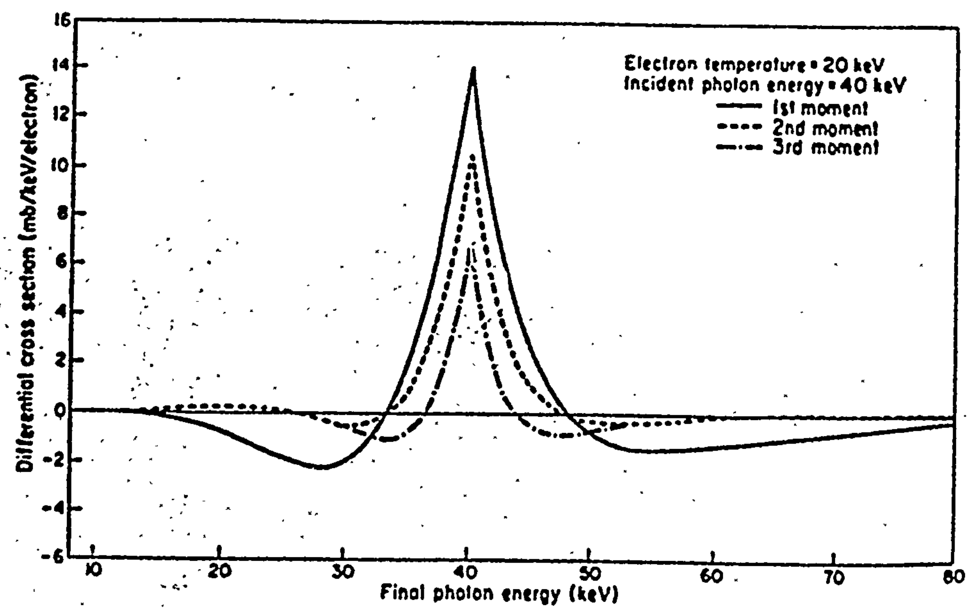
The differential scattering cross section: $n = 0$; $T = 1$.



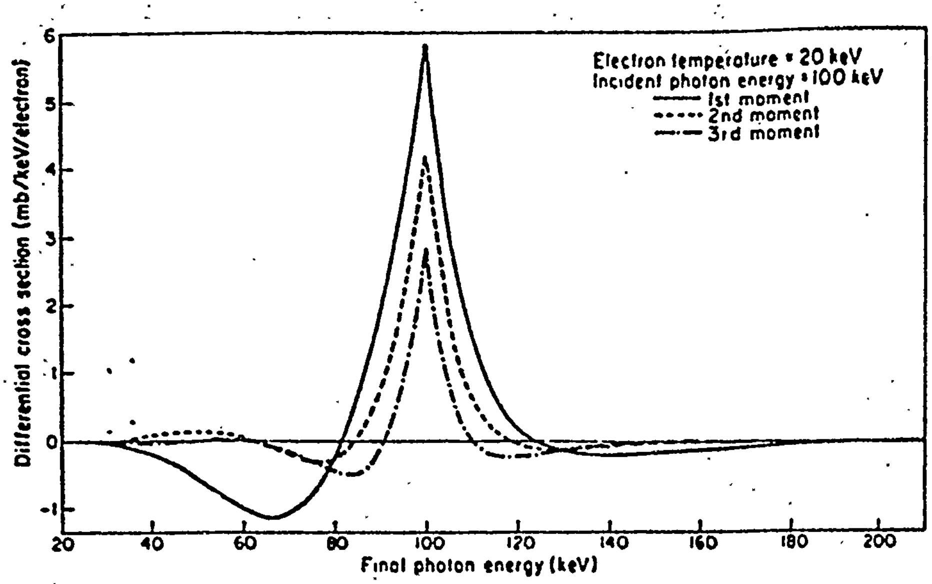
The differential scattering cross section: $n = 0$; $T = 20$.



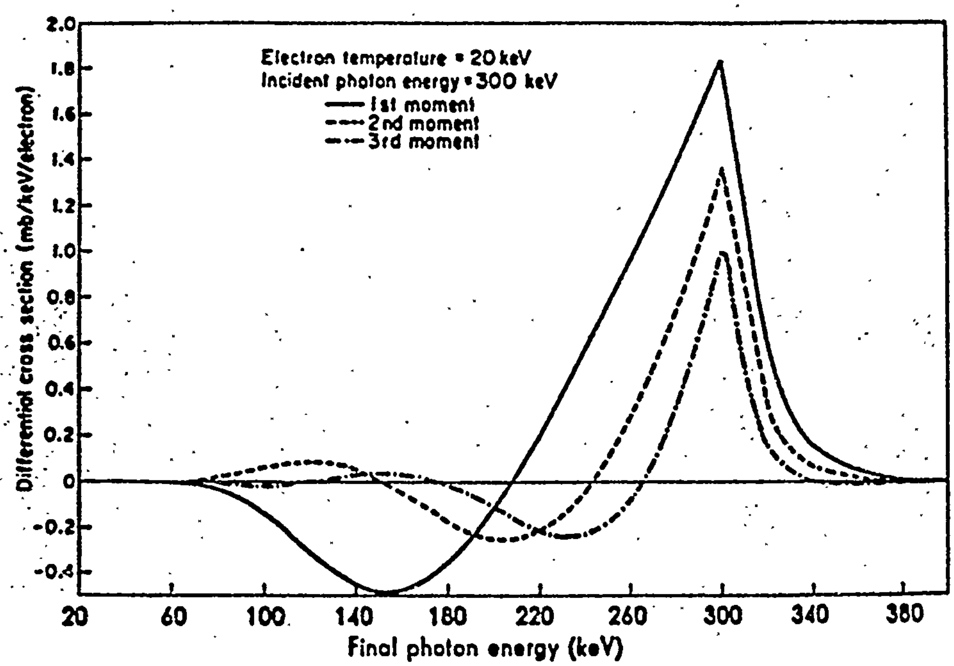
The differential scattering cross section: $n = 0$; $T = 20$.



The differential scattering cross section: $n = 1, 2, 3$; $T = 20$.



The differential scattering cross section: $n = 1, 2, 3$; $T = 20$.



The differential scattering cross section: $n = 1, 2, 3$; $T = 20$.

D. The Fokker-Planck Treatment of Compton and Inverse Compton Scattering

The scattering kernel for Compton and inverse Compton effects just derived, while accurately describing the physics of photon scattering from a Maxwellian gas of free electrons, is rather complex in that is defined in terms of a multiple integral [see Eq. (676)]. In addition, the use of this kernel in the equation of transfer introduces further integrals over frequency and angle [see Eq. (147)]. We now describe a simplification of this scattering description which leads to the elimination of all integrals defining the scattering kernel as well as the integral over frequency in the equation of transfer. The assumption required to achieve this result (the Fokker-Planck approximation) is that the electron temperatures and photon frequencies (measured as energies) are small compared to the rest energy of an electron.

In order to effect this simplification, we first expand $I(\nu',\vec{\Omega}')$ in Eq. (147) in a Taylor series about $\nu' = \nu$, i.e.,

$$I(\nu',\vec{\Omega}') = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n I(\nu,\vec{\Omega}')}{\partial \nu^n} (\nu' - \nu)^n \quad . \tag{684}$$

Inserting Eq. (684) into Eq. (147) and integrating term by term over ν', we obtain

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} I(\nu,\vec{\Omega}) = \sigma_a'(\nu)\left[B(\nu) - I(\nu,\vec{\Omega})\right]$$

$$+ \int_{4\pi} d\vec{\Omega}' \sum_{n=0}^{\infty} N_n(\nu,\vec{\Omega}\cdot\vec{\Omega}')\nu^n \frac{\partial^n I(\nu,\vec{\Omega}')}{\partial \nu^n} - \sigma_s(\nu) I(\nu,\vec{\Omega})$$

$$+ \frac{c^2}{2h\nu^3} I(\nu,\vec{\Omega}) \int_{4\pi} d\vec{\Omega}' \sum_{n=0}^{\infty} M_n(\nu,\vec{\Omega}\cdot\vec{\Omega}')\nu^n \frac{\partial^n I(\nu,\vec{\Omega}')}{\partial \nu^n} \quad , \tag{685}$$

where we have defined

$$\sigma_s(\nu) = \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \ \sigma_s(\nu+\nu', \vec{\Omega}\cdot\vec{\Omega}') \quad , \tag{686}$$

$$N_n(\nu, \vec{\Omega}\cdot\vec{\Omega}') = \frac{1}{n!} \int_0^\infty d\nu' \ \frac{\nu}{\nu'} \left(\frac{\nu' - \nu}{\nu}\right)^n \sigma_s(\nu'+\nu, \vec{\Omega}'\cdot\vec{\Omega}) \quad , \tag{687}$$

$$M_n(\nu, \vec{\Omega}\cdot\vec{\Omega}') = \frac{1}{n!} \int_0^\infty d\nu' \ \left(\frac{\nu' - \nu}{\nu}\right)^n$$

$$\cdot \left[\frac{\nu}{\nu'} \sigma_s(\nu'+\nu, \vec{\Omega}\cdot\vec{\Omega}') - \left(\frac{\nu}{\nu'}\right)^3 \sigma_s(\nu+\nu', \vec{\Omega}\cdot\vec{\Omega}')\right] . \tag{688}$$

The formal Taylor series expansion has converted the integral operator in frequency in Eq. (147) into an infinite order differential operator. The usefulness of this procedure is that for small electron temperature

$$\alpha \equiv \frac{kT}{m_o c^2} << 1 \quad , \tag{689}$$

and small photon energies

$$\gamma \equiv \frac{h\nu}{m_o c^2} << 1 \quad , \tag{690}$$

this infinite order operator effectively truncates itself to one of finite order. In particular, to first order in α and γ, we have the explicit results

$$\sigma_s(\nu) = \sigma_{Th}(1 - 2\gamma) \quad . \tag{691}$$

$$N_o(\nu, \vec{\Omega}\cdot\vec{\Omega}') = \frac{3}{16\pi} \sigma_{Th}\left[(1 - \gamma + 2\alpha) + (\vec{\Omega}\cdot\vec{\Omega}')(\gamma - 4\alpha)\right.$$

$$\left. + (\vec{\Omega}\cdot\vec{\Omega}')^2(1 - \gamma - 6\alpha) + (\vec{\Omega}\cdot\vec{\Omega}')^3(\gamma + 4\alpha)\right] , \tag{692}$$

$$N_1(\nu,\hat{\Omega}\cdot\hat{\Omega}') = \frac{3}{16\pi}\,\sigma_{Th}[1 - (\hat{\Omega}\cdot\hat{\Omega}') + (\hat{\Omega}\cdot\hat{\Omega}')^2 - (\hat{\Omega}\cdot\hat{\Omega}')^3](\gamma-2\alpha) \quad , \tag{693}$$

$$N_2(\nu,\hat{\Omega}\cdot\hat{\Omega}') = \frac{3}{16\pi}\,\sigma_{Th}[1 - (\hat{\Omega}\cdot\hat{\Omega}') + (\hat{\Omega}\cdot\hat{\Omega}')^2 - (\hat{\Omega}\cdot\hat{\Omega}')^3](\alpha) \quad , \tag{694}$$

$$M_o(\nu,\hat{\Omega},\cdot\hat{\Omega}') = -\frac{3}{16\pi}\,\sigma_{Th}[1 - (\hat{\Omega}\cdot\hat{\Omega}') + (\hat{\Omega}\cdot\hat{\Omega}')^2 - (\hat{\Omega}\cdot\hat{\Omega}')^3](2\gamma) \quad , \tag{695}$$

$$M_1(\nu,\hat{\Omega}\cdot\hat{\Omega}') = \frac{3}{16\pi}\,\sigma_{Th}[1 - (\hat{\Omega}\cdot\hat{\Omega}') + (\hat{\Omega}\cdot\hat{\Omega}')^2 - (\hat{\Omega}\cdot\hat{\Omega}')^3](2\gamma) \quad . \tag{696}$$

Here σ_{Th} is the Thompson scattering cross section given by

$$\sigma_{Th} = \frac{8\pi}{3}\,Nr_o^2 \quad . \tag{697}$$

All of the other $N_n(\nu)$ and $M_n(\nu)$ are of higher order in α and γ. Introducing these results into Eq. (685) and employing a somewhat more compact notation, we can write the equation of transfer with scattering described to first order in α and γ as

$$\frac{1}{c}\frac{\partial I(\nu,\hat{\Omega})}{\partial t} + \hat{\Omega}\cdot\vec{\nabla}I(\nu,\hat{\Omega}) = \sigma_a'(\nu)[B(\nu) - I(\nu,\hat{\Omega})]$$

$$- \sigma_{Th}(1 - 2\gamma)I(\nu,\hat{\Omega}) + \sigma_{Th}\int_{4\pi} d\hat{\Omega}' \sum_{n=0}^{3}\left(\frac{2n+1}{4\pi}\right)P_n(\hat{\Omega}\cdot\hat{\Omega}')S_nI(\nu,\hat{\Omega}')$$

$$- \frac{3\sigma_{Th}}{16\pi}\frac{c^2}{2h\nu^3}\,\gamma I(\nu,\hat{\Omega})\left(1 - \nu\frac{\partial}{\partial\nu}\right) \cdot$$

$$\cdot \int_{4\pi} d\hat{\Omega}'[1 - (\hat{\Omega}\cdot\hat{\Omega}') + (\hat{\Omega}\cdot\hat{\Omega}')^2 - (\hat{\Omega}\cdot\hat{\Omega}')^3]I(\nu,\hat{\Omega}') \quad . \tag{698}$$

Here $P_n(z)$ is the usual n^{th} order Legendre polynomial and the scattering operators S_n are defined as

$$S_o = [1 - \gamma(1 - \nu \frac{\partial}{\partial \nu}) - \alpha(2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2})] \quad , \tag{699}$$

$$S_1 = \frac{2}{5} [\gamma(1 - \nu \frac{\partial}{\partial \nu}) - \alpha(1 - 2\nu \frac{\partial}{\partial \nu} + \nu^2 \frac{\partial^2}{\partial \nu^2})] \quad , \tag{700}$$

$$S_2 = \frac{1}{10} [1 - \gamma(1 - \nu \frac{\partial}{\partial \nu}) - \alpha(6 - 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2})] \quad , \tag{701}$$

$$S_3 = \frac{3}{70} [\gamma(1 - \nu \frac{\partial}{\partial \nu}) + \alpha(4 + 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2})] \quad . \tag{702}$$

Equation (698) is the result of the formal expansion of the scattering operator to first order in α and γ. We see that the integral operator in the exact description of scattering has been replaced by a second order differential operator. It can be verified by direct substitution that the small α and γ expansion has not destroyed the equilibrium solution of the scattering operator. That is, with the induced (quadratic in I) scattering terms retained, the equilibrium solution of Eq. (698) is the Planck function, and with the neglect of the induced terms the Wien distribution, Eq. (665), is the equilibrium solution.

Equation (698) can be simplified substantially without introducing any further assumptions. A straightforward way to effect this simplification is to consider Eq. (698) projected onto the basis elements of a spherical harmonic function space. For simplicity we work in plane geometry (the arguments are equally valid in general geometry) and momentarily neglect the induced scattering (nonlinear terms). We then have from Eq. (698), generalizing the Legendre polynomial expansion which led to Eqs. (565) and (566) to anisotropic scattering,

$$\frac{1}{c}\frac{\partial I_o}{\partial t} + \frac{\partial I_1}{\partial z} + \sigma_a'(I_o - 4\pi B) + \sigma_{Th}(1 - 2\gamma - S_o)I_o = 0 \quad , \tag{703}$$

$$\left(\frac{2n+1}{c}\right) \frac{\partial I_n}{\partial t} + n \frac{\partial I_{n-1}}{\partial z} + (n+1) \frac{\partial I_{n+1}}{\partial z}$$

$$+ (2n+1)\left[\sigma'_a + \sigma_{Th}(1 - 2\gamma - S_n)\right] I_n = 0 \quad ,$$

$$1 \leq n \leq 3 \quad , \qquad (704)$$

$$\left(\frac{2n+1}{c}\right) \frac{\partial I_n}{\partial t} + n \frac{\partial I_{n-1}}{\partial z} + (n+1) \frac{\partial I_{n+1}}{\partial z}$$

$$+ (2n+1)\left[\sigma'_a + \sigma_{Th}(1 - 2\gamma)\right] I_n = 0 \quad ,$$

$$n \geq 4 \quad , \qquad (705)$$

where

$$I_n = 2\pi \int_{-1}^{1} d\mu P_n(\mu) I(z,\nu,\mu,t) \quad . \qquad (706)$$

We focus our attention on the scattering terms in Eqs. (704) through (706). In Eqs. (704) and (705) we make the replacements

$$\sigma_{Th}(1 - 2\gamma - S_1) \rightarrow \sigma_{Th} \quad ,$$

$$\sigma_{Th}(1 - 2\gamma - S_2) \rightarrow \frac{9}{10}\, \sigma_{Th} \quad ,$$

$$\sigma_{Th}(1 - 2\gamma - S_3) \rightarrow \sigma_{Th} \quad ,$$

$$\sigma_{Th}(1 - 2\gamma) \rightarrow \sigma_{Th} \quad . \qquad (707)$$

The justification is that each term on the left hand side of Eq. (707) has a dominant zeroth order (in γ and α) term, which we retain, and first order terms, which we neglect, compared to the zeroth order terms. We note, however, that we cannot make a similar simplification in Eq. (703) since the term $(1 - 2\gamma - S_o)$ is of order α and γ, rather than of order unity (or 9/10), as are the similar terms in Eqs. (704) and (705). Introducing Eq. (707) into Eqs. (704) and (705), we find that Eqs. (703) through (705) are the spherical harmonic projections of the equation of transfer, reverting back to general geometry.

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \sigma_a'(\nu)[B(\nu) - I(\nu,\vec{\Omega})]$$

$$- \sigma_{Th} I(\nu,\vec{\Omega}) + \frac{3\sigma_{Th}}{16\pi}\int_{4\pi} d\vec{\Omega}'[1 + (\vec{\Omega}\cdot\vec{\Omega}')^2]I(\nu,\vec{\Omega}')$$

$$+ \frac{\sigma_{Th}}{4\pi}\int_{4\pi} d\vec{\Omega}'\left[\alpha\nu^2 \frac{\partial^2}{\partial\nu^2} + (\gamma - 2\gamma)\nu\frac{\partial}{\partial\nu} + \gamma\right]I(\nu,\vec{\Omega}') \quad . \qquad (708)$$

To Eq. (708) we need add the contribution of the nonlinear induced scattering terms in Eq. (698). Since these terms are of order γ, they can be neglected in all but the zeroth angular moment of the equation of transfer, just as we neglected all terms of order α and γ in the linear analysis just completed except in the zeroth angular moment relationship. This implies the replacement in Eq. (698)

$$I(\nu,\vec{\Omega})\left(1 - \nu\frac{\partial}{\partial\nu}\right)\int_{4\pi} d\vec{\Omega}'[1 - (\vec{\Omega}\cdot\vec{\Omega}') + (\vec{\Omega}\cdot\vec{\Omega}')^2 - (\vec{\Omega}\cdot\vec{\Omega}')^3]I(\nu,\vec{\Omega}')$$

$$+ \frac{1}{4\pi}\int_{4\pi} d\vec{\Omega}I(\nu,\vec{\Omega})\left(1 - \nu\frac{\partial}{\partial\nu}\right) \cdot$$

$$\cdot \int_{4\pi} d\vec{\Omega}'[1 - (\vec{\Omega}\cdot\vec{\Omega}') + (\vec{\Omega}\cdot\vec{\Omega}')^2 - (\vec{\Omega}\cdot\vec{\Omega}')^3]I(\nu,\vec{\Omega}') \quad . \qquad (709)$$

Thus, the full form of Eq. (708), including the effects of induced scattering, is

$$\frac{1}{c}\frac{\partial I(\nu,\vec{\Omega})}{\partial t} + \vec{\Omega}\cdot\vec{\nabla}I(\nu,\vec{\Omega}) = \sigma_a'(\nu)\left[B(\nu) - I(\nu,\vec{\Omega})\right]$$

$$- \sigma_{Th}I(\nu,\vec{\Omega}) + \frac{3\sigma_{Th}}{16\pi}\int_{4\pi} d\vec{\Omega}'\left[1 + (\vec{\Omega}\cdot\vec{\Omega}')^2\right]I(\nu,\vec{\Omega}')$$

$$+ \frac{\sigma_{Th}}{4\pi}\int_{4\pi} d\vec{\Omega}'\left[\alpha\nu^2\frac{\partial^2}{\partial\nu^2} + (\gamma - 2\alpha)\nu\frac{\partial}{\partial\nu} + \gamma\right]I(\nu,\vec{\Omega}')$$

$$- \frac{3\sigma_{Th}}{64\pi^2}\frac{c^2}{h\nu^3}\gamma\int_{4\pi} d\vec{\Omega}' I(\nu,\vec{\Omega}')\int_{4\pi} d\vec{\Omega}'' \cdot$$

$$\cdot \left[1 - (\vec{\Omega}\cdot\vec{\Omega}') + (\vec{\Omega}\cdot\vec{\Omega}')^2 - (\vec{\Omega}\cdot\vec{\Omega}')^3\right]\left(1 - \nu\frac{\partial}{\partial\nu}\right)I(\nu,\vec{\Omega}'') \quad , \quad (710)$$

which is simplified, but apriori just as accurate form of Eq. (698). In particular, Eq. (710) contains far fewer scattering terms than does Eq. (698), and the terms which account for energy transfer in the scattering interaction, i.e., those proportional to α and γ, are isotropic in Eq. (710), whereas they are angularly dependent in Eq. (698). Since the zeroth angular moments of Eqs. (698) and (710) are identical, they yield the same equilibrium solution, namely a Planck function at the material temperature.

REFERENCES

A. General

Much of the material in these notes was taken from the book:

G.C. Pomraning, The Equations of Radiation Hydrodynamics, Pergamon Press, Oxford (1973).

References for this material and general reading references can be found in this book. A very readable introduction to the subject of radiation hydrodynamics is:

P.M. Cambell, "An Introduction to High Temperature Radiation Gas Dynamics," Air Force Weapons Laboratory Report AFWL-TR-69-10 (1969).

The classic book in radiative transfer is:

S. Chandrasekhar, Radiative Transfer, Dover, New York (1960).

B. New Material

Some new material not found in any of the above references was included in the class and in these notes. Some references for this material are given below.

1. Transport in a Vacuum

This material is discussed in any good heat transfer book dealing with radiative transfer. A particularly easy to read treatment which follows the notes quite closely is found in the radiative transfer chapter of:

R.B. Bird, W.E. Stewart, and E.N. Lightfoot, Transport Phenomena, Wiley, New York, (1960).

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